Good gradings of basic Lie superalgebras

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Abstract

We classify good $\mathbb{Z}$-gradings of basic Lie superalgebras over an algebraically closed field $\mathbb{F}$ of characteristic zero. Good $\mathbb{Z}$-gradings are used in quantum Hamiltonian reduction for affine Lie superalgebras, where they play a role in the construction of super W-algebras. We also describe the centralizer of a nilpotent even element and of an $\mathfrak{sl}_2$-triple in $\mathfrak{gl}(m|n)$ and $\mathfrak{osp}(m|2n)$.

0 Introduction

Good $\mathbb{Z}$-gradings of basic Lie superalgebras are used in the construction of super W-algebras, both finite and affine [5]. In this paper, we classify good $\mathbb{Z}$-gradings of basic Lie superalgebras. A finite-dimensional simple Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is called basic if $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra and there exists an even nondegenerate invariant bilinear form on $\mathfrak{g}$. A $\mathbb{Z}$-grading $\mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}(j)$ is called good if there exists $e \in \mathfrak{g}_{\bar{0}}(2)$ such that the map ad $e : \mathfrak{g}(j) \to \mathfrak{g}(j+2)$ is injective for $j \leq -1$ and surjective for $j \geq -1$. If a $\mathbb{Z}$-grading of $\mathfrak{g}$ is defined by a semisimple element $h \in \mathfrak{g}_{\bar{0}}$, then this condition is equivalent to all of the eigenvalues of $\mathrm{ad}(h)$ on the centralizer $\mathfrak{g}^e$ of e in $\mathfrak{g}$ being non-negative.

An example of a good $\mathbb{Z}$-grading for a nilpotent element $e \in \mathfrak{g}_{\bar{0}}$ is the Dynkin grading. By the Jacobson-Morosov Theorem, e belongs to an $\mathfrak{sl}_2$-triple $\mathfrak{s} = \{e, f, h\} \subset \mathfrak{g}_{\bar{0}}$, where $[e, f] = h$, $[h, e] = 2e$ and $[h, f] = -2f$. By $\mathfrak{sl}_2$ theory, the grading of $\mathfrak{g}$ defined by ad h is a good $\mathbb{Z}$-grading for e.

Affine W-algebras are vertex algebras which can be realized using the homology of a BRST complex of a simple finite-dimensional Lie superalgebra $\mathfrak{g}$ with a non-degenerate even supersymmetric invariant bilinear form. If x is an ad-diagonalizable element of $\mathfrak{g}$ with half integer eigenvalues and if f is an even nilpotent element of $\mathfrak{g}$ such that $[x, f] = -f$ and the eigenvalues of $\mathrm{ad}(x)$ on the centralizer $\mathfrak{g}^f$ of f in $\mathfrak{g}$ are all non-positive, then for each complex number k, one can define a vertex algebra $W^k(\mathfrak{g}, x, f)$, as was shown by Kac, Roan and Wakimoto in 2003 [11].

The minimal W-algebras $W^k(\mathfrak{g}, x, f_\theta)$, where f_θ is a root vector of the lowest root θ (which is assumed to be even), have been studied more extensively [10, 11]. This class of W-algebras contains the well known superconformal algebras. Let $\widehat{\mathfrak{g}}$ be the (non-twisted) affinization of $\mathfrak{g}$ and let O_k be the BGG-category of $\widehat{\mathfrak{g}}$ at level k. A functor H from the category O_k to the category of integer graded modules of $W^k(\mathfrak{g}, x, f_\theta)$ was given by Kac, Roan and Wakimoto in [11]. The quantum reduction functor has many nice properties, allowing one to transfer information between the two

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categories of modules. In particular, in 2005, Arakawa proved that this functor is exact and that the image of a simple highest weight module is either zero or irreducible [1].

A finite W-algebra is defined as follows [13, 16, 7]. Given a good $\mathbb{Z}$-grading $\mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}(j)$ for a nilpotent element $e \in \mathfrak{g}(2)$, choose an isotropic subspace $\mathfrak{l}$ of $\mathfrak{g}(-1)$ with respect to the skew-supersymmetric bilinear form defined by $\omega(x, y) = (e, [x, y])$. Let $\mathfrak{m} = \mathfrak{l} \oplus \bigoplus_{j\leq -2}\mathfrak{g}(j)$ and $\mathfrak{n} = \mathfrak{l}^{\perp} \oplus \bigoplus_{j\leq -2}\mathfrak{g}(j)$, where $\mathfrak{l}^{\perp}$ is the complement of $\mathfrak{l}$ with respect to ω. Define $\chi : \mathfrak{m} \to \mathbb{C}$ by $\chi(x) = (x, e)$, and let $\mathcal{Q} = U(\mathfrak{g}) \otimes_{U(\mathfrak{m})} \mathbb{C}_{\chi}$. The (super) *finite W-algebra* associated to e for this choice of grading and isotropic subspace $\mathfrak{l}$ is defined to be $W^{fin}(\mathfrak{g}, e) = \mathcal{Q}^{\mathrm{ad}\ n}$. The algebra structure of $W^{fin}(\mathfrak{g}, e)$ is induced from that of $U(\mathfrak{g})$.

Good $\mathbb{Z}$-gradings of simple finite-dimensional Lie algebras were classified by A.G. Elashvili and V.G. Kac in [6]. K. Baur and N. Wallach classified nice parabolic subalgebras of reductive Lie algebras in [2], which correspond to good even $\mathbb{Z}$-gradings by [6, Theorem 2.1]. J. Brundan and S. Goodwin classified good $\mathbb{R}$-gradings of semisimple Lie algebras in [3], and proved that the isomorphism type of a (non-super) finite W-algebra does not depend on the choice of good grading. W.L. Gan and V. Ginzburg proved that a (non-super) finite W-algebra does not depend on the choice of the isotropic subspace $\mathfrak{l}$ [7].

The paper is organized as follows. In Section 2, we study $\mathbb{Z}$-gradings of basic Lie superalgebras. We obtain a criterion for when two diagram characteristics determine the same $\mathbb{Z}$-grading by using the action of the Weyl groupoid. In Section 3, we describe explicitly the centralizers of nilpotent even elements and of $\mathfrak{sl}_2$-triples in $\mathfrak{gl}(m|n)$ and $\mathfrak{osp}(m|2n)$. In Section 4, we establish some general results for good $\mathbb{Z}$-gradings of basic Lie superalgebras. We also examine the question of extending good $\mathbb{Z}$-gradings from $\mathfrak{g}_{\bar{0}}$ to $\mathfrak{g}$. In Section 5, we prove that all good $\mathbb{Z}$-gradings of the exceptional Lie superalgebras $F(4)$, $G(3)$, and $D(2, 1, \alpha)$ are Dynkin gradings. In Sections 6, 7 and 8, we classify the good $\mathbb{Z}$-gradings of $\mathfrak{psl}(2|2)$, $\mathfrak{gl}(m|n)$, and $\mathfrak{osp}(m|2n)$, respectively. In particular, for each nilpotent even element (up to conjugacy) we describe all $\mathbb{Z}$-gradings for which the element is good.

We classify the good $\mathbb{Z}$-gradings of $\mathfrak{gl}(m|n)$ and $\mathfrak{osp}(m|2n)$ using pyramids. Pyramids were defined in [3, 6] to describe the good $\mathbb{Z}$-gradings of $\mathfrak{gl}(n)$, $\mathfrak{so}(m)$ and $\mathfrak{sp}(2n)$. We generalize these definitions to the Lie superalgebras $\mathfrak{gl}(m|n)$ and $\mathfrak{osp}(m|2n)$. For $\mathfrak{gl}(m|n)$, a symmetric pyramid is defined for each nilpotent even element e essentially by taking an $\mathfrak{sl}_2$-triple $\mathfrak{s}$ containing e and then looking at the $\mathfrak{sl}_2$-strings in the standard representation of $\mathfrak{s}$. One arranges rows of boxes in the upper half plane such that each row corresponds to an $\mathfrak{sl}_2$-string, the rows have non-increasing length in the positive y direction, and the left coordinate of each box equals the weight of the vector to which it corresponds. Then by $\mathfrak{sl}_2$ theory this pyramid is symmetric about the y-axis. Any pyramid for $\mathfrak{gl}(m|n)$ can be obtained from a symmetric pyramid by shifting the rows horizontally. For $\mathfrak{osp}(m|2n)$, one adjusts the symmetric pyramid to contain a central symmetry about the origin.

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1 Preliminaries

We begin with recalling the definitions of $\mathfrak{gl}(m|n)$ and $\mathfrak{sl}(m|n)$. Let $M_{r,s}$ denote the set of $r \times s$ matrices. As a vector space $\mathfrak{gl}(m|n)$ is $M_{m+n,m+n}$, where

$$\mathfrak{g}_{\bar{0}} = \left\{ \left(\begin{array}{cc} A & 0 \\ 0 & B \end{array} \right) \mid A \in M_{m,m},\ B \in M_{n,n} \right\}$$
$$\mathfrak{g}_{\bar{1}} = \left\{ \left(\begin{array}{cc} 0 & C \\ D & 0 \end{array} \right) \mid C \in M_{m,n},\ D \in M_{n,m} \right\}.$$

The bracket operation is defined on homogeneous elements as follows: if $X \in \mathfrak{g}_i$, $Y \in \mathfrak{g}_j$, then $[X,Y] := XY - (-1)^{ij}YX$, and it is extended linearly to the superalgebra. The Lie superalgebra $\mathfrak{sl}(m|n)$ is defined to be

$$\mathfrak{sl}(m|n) = \left\{ X = \left(\begin{array}{cc} A & C \\ D & B \end{array} \right) \in \mathfrak{gl}(m|n) \mid \mathrm{supertr}(X) := \mathrm{tr}(A) - \mathrm{tr}(B) = 0 \right\}.$$

1.1 Basic Lie superalgebras

Finite-dimensional simple Lie superalgebras were classified by V.G. Kac in [9]. These can be separated into three types: basic, strange and Cartan. A finite-dimensional simple Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is called *basic* if $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra and $\mathfrak{g}$ has an even nondegenerate invariant bilinear form $(\cdot,\cdot)$. This form is necessarily supersymmetric. These are: $\mathfrak{sl}(m|n) : m \neq n$, $\mathfrak{psl}(n|n) := \mathfrak{sl}(n|n)/\langle I_{2n} \rangle$, $\mathfrak{osp}(m|n)$, $D(2,1,\alpha)$, $F(4)$, $G(3)$, and finite dimensional simple Lie algebras.

Table 1

$\mathfrak{g}$		$\mathfrak{g}_{\bar{0}}$	$\mathfrak{Z}(\mathfrak{g}_{\bar{0}})$	κ
$\mathfrak{sl}(m\|n)$	$m,n \geq 1,\ m \neq n$	$\mathfrak{sl}(m) \times \mathfrak{sl}(n) \times \mathbb{C}$	$\mathbb{C}$	
$\mathfrak{psl}(n\|n)$	$n \geq 1$	$\mathfrak{sl}(n) \times \mathfrak{sl}(n)$	$\{0\}$	0
$\mathfrak{osp}(2\|2n)$	$n \geq 1$	$\mathbb{C} \times \mathfrak{sp}(2n)$	$\mathbb{C}$	
$\mathfrak{osp}(2n+2\|2n)$	$n \geq 1$	$\mathfrak{so}(2n+2) \times \mathfrak{sp}(2n)$	$\{0\}$	0
$\mathfrak{osp}(m\|2n)$	$m,n \geq 1,\ m \neq 2, 2n+2$	$\mathfrak{so}(m) \times \mathfrak{sp}(2n)$	$\{0\}$	
$D(2,1,\alpha)$	$\alpha \neq 0, -1$	$\mathfrak{sl}(2) \times \mathfrak{sl}(2) \times \mathfrak{sl}(2)$	$\{0\}$	0
$F(4)$		$\mathfrak{so}(7) \times \mathfrak{sl}(2)$	$\{0\}$	
$G(3)$		$G_2 \times \mathfrak{sl}(2)$	$\{0\}$	

Note that if $\mathfrak{g}$ is a finite-dimensional simple Lie superalgebra, then $\mathfrak{g}_{\bar{0}}$ is a reductive if and only if the representation of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ is completely reducible [17]. The Lie superalgebras $\mathfrak{sl}(m|n)$, $\mathfrak{osp}(m|2n)$, $D(2,1,\alpha)$, $F(4)$ and $G(3)$ are Kac-Moody superalgebras, i.e. they are defined by their Cartan matrix [9].

Let $\mathfrak{g}$ be a basic Lie superalgebra. Elements of $\mathfrak{g}_{\bar{0}}$ are called *even*, while elements of $\mathfrak{g}_{\bar{1}}$ are called *odd*. We can write $\mathfrak{g}_{\bar{0}} = \mathfrak{g}'_{\bar{0}} \times \mathfrak{Z}(\mathfrak{g}_{\bar{0}})$, where $\mathfrak{Z}(\mathfrak{g}_{\bar{0}})$ is the center of $\mathfrak{g}_{\bar{0}}$ and $\mathfrak{g}'_{\bar{0}} := [\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{0}}]$ is semisimple. If $\mathfrak{g} \neq \mathfrak{psl}(n|n), \mathfrak{osp}(2n+2|2n), D(2,1,\alpha)$ then the Killing form $\kappa(x,y) := \text{supertr}((\text{ad } x)(\text{ad } y))$ is nondegenerate, and otherwise it is identically zero [9].

For each $x \in \mathfrak{g}_{\bar{0}}$ the map $\exp(\text{ad } x)$ is an automorphism of $\mathfrak{g}$. The group G generated by these automorphisms is called the *group of inner automorphisms* of $\mathfrak{g}$. Every inner automorphism of $\mathfrak{g}_{\bar{0}}$ extends to an inner automorphism of $\mathfrak{g}$ [9].

1.2 Decompositions of $\mathfrak{g}$

A subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is called a *Cartan subalgebra* if $\mathfrak{h}$ is nilpotent and $\mathfrak{h}$ equals its normalizer in $\mathfrak{g}$. If $\mathfrak{g}$ is a basic Lie superalgebra, then $\mathfrak{h}$ is a Cartan subalgebra for $\mathfrak{g}$ if and only if it is a Cartan subalgebra for $\mathfrak{g}_{\bar{0}}$. All Cartan subalgebras of $\mathfrak{g}$ are conjugate, because they are conjugate in the reductive Lie algebra $\mathfrak{g}_{\bar{0}}$.

Fix a Cartan subalgebra $\mathfrak{h}$. For $\alpha \in \mathfrak{h}^*$, let $\mathfrak{g}_\alpha := \{x \in \mathfrak{g} \mid [h,x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}$ and let $\Delta = \{\alpha \in \mathfrak{h}^* \setminus \{0\} \mid \mathfrak{g}_\alpha \neq 0\}$. Then $\mathfrak{g}_0 = \mathfrak{h}$ and $\mathfrak{g}$ has a root space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$. The $\mathbb{Z}/2\mathbb{Z}$-grading of $\mathfrak{g}$ determines a decomposition of Δ into the disjoint union of the even roots $\Delta_{\bar{0}}$ and the odd roots $\Delta_{\bar{1}}$. Let W denote the Weyl group of $\Delta_{\bar{0}}$. Then $\Delta_{\bar{0}}$ and $\Delta_{\bar{1}}$ are invariant under W.

An element $h \in \mathfrak{h}$ is called *regular* if $\text{Re } \alpha(h) \neq 0$ for all $\alpha \in \Delta$. A regular element $h \in \mathfrak{h}$ determines a decomposition of the roots $\Delta = \Delta^+ \sqcup \Delta^-$ where $\Delta^+ := \{\alpha \in \Delta \mid \text{Re } \alpha(h) > 0\}$ and $\Delta^- := \{\alpha \in \Delta \mid \text{Re } \alpha(h) < 0\}$. This then determines a decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ where $\mathfrak{n}^+ := \oplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$ and $\mathfrak{n}^- := \oplus_{\alpha \in \Delta^-} \mathfrak{g}_\alpha$. Such a decomposition is called a *triangular decomposition* [15]. We have an induced triangular decomposition $\mathfrak{g}_{\bar{0}} = \mathfrak{n}^-_{\bar{0}} \oplus \mathfrak{h} \oplus \mathfrak{n}^+_{\bar{0}}$ given by $\Delta_{\bar{0}} = \Delta^+_{\bar{0}} \sqcup \Delta^-_{\bar{0}}$. Corresponding to a decomposition $\Delta = \Delta^+ \sqcup \Delta^-$, a *base* is a set of simple roots $\Pi \subset \Delta^+$ (resp. $\Pi_{\bar{0}} \subset \Delta^+_{\bar{0}}$) for $\mathfrak{g}$ (resp. $\mathfrak{g}_{\bar{0}}$).

A subalgebra $\mathfrak{b} \subset \mathfrak{g}$ is called a Borel subalgebra if $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ for some triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. Since $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra, the group of inner automorphisms of $\mathfrak{g}_{\bar{0}}$ acts transitively on the set of Borel subalgebras for $\mathfrak{g}_{\bar{0}}$. Since every inner automorphism of $\mathfrak{g}_{\bar{0}}$ extends to an inner automorphism of $\mathfrak{g}$, the Borel subalgebras of $\mathfrak{g}_{\bar{0}}$ are conjugate in $\mathfrak{g}$.

1.3 The bilinear form

Let $\mathfrak{g}$ be a basic Lie superalgebra, and let $(\cdot,\cdot)$ be a nondegenerate invariant even supersymmetric bilinear form on $\mathfrak{g}$. Such a form is unique up to multiplication by a scalar [9]. There is an invariant even supersymmetric bilinear form on $\mathfrak{gl}(m|n)$, which when restricted to $\mathfrak{sl}(m|n)$ has kernel equal to the center of $\mathfrak{sl}(m|n)$ [9]. We will also denote this form by $(\cdot,\cdot)$. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra, and let Δ be the set of roots.

Theorem 1.1 (V.G. Kac [9]). *If $\mathfrak{g}$ is a basic Lie superalgebra or if $\mathfrak{g} = \mathfrak{gl}(m|n)$ or $\mathfrak{sl}(n|n)$, then*

(i) $(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ *if $\alpha \neq -\beta$ for $\alpha, \beta \in \Delta \cup \{0\}$;*

(ii) $(\cdot,\cdot)$ *determines a nondegenerate pairing of $\mathfrak{g}_\alpha$ with $\mathfrak{g}_{-\alpha}$ for $\alpha \in \Delta$;*

(iii) if $\mathfrak{g} \neq \mathfrak{gl}(m|n), \mathfrak{sl}(n|n)$, then the restriction of $(\cdot,\cdot)$ to $\mathfrak{h}$ is nondegenerate;

(iv) if $\mathfrak{g} = \mathfrak{sl}(n|n)$, then the kernel of $(\cdot,\cdot)$ equals the center of $\mathfrak{g}$;

(v) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \neq 0$ *if and only if* $\alpha, \beta, \alpha+\beta \in \Delta \cup \{0\}$*;*

(vi) if $\mathfrak{g} \neq \mathfrak{psl}(2|2)$*, then dim* $\mathfrak{g}_\alpha = 1$ *for all* $\alpha \in \Delta$*.*

If we fix $(\cdot,\cdot)$, then we can use $(\cdot,\cdot)$ to identify $\mathfrak{h}$ with $\mathfrak{h}^\star$. Then $(\cdot,\cdot)$ is defined on $\Delta \subset \mathfrak{h}^\star$ through this identification. A root $\alpha \in \Delta$ is called *isotropic* if $(\alpha,\alpha) = 0$. For a basic Lie superalgebra, a simple isotropic root is necessarily odd.

1.4 Even and odd reflections

We recall the notion of odd reflections for basic Lie superalgebras [14].

Two Borel subsuperalgebras $\mathfrak{b}, \mathfrak{b}' \subset \mathfrak{g}$ are connected by an *odd reflection* along α_k if and only if α_k is a simple odd isotropic root of $\mathfrak{b}$ and

$$\Delta'^+ = \big(\Delta^+ \setminus \{\alpha_k\}\big) \cup \{-\alpha_k\}. \tag{1}$$

For the bases $\Pi \subset \Delta^+$ and $\Pi' \subset \Delta'^+$, we say that Π' is obtained from Π by an odd reflection with respect to α_k. This is defined explicitly on Π by

$$r_k(\alpha_i) := \begin{cases} -\alpha_k, & \text{if } i = k; \\ \alpha_i, & \text{if } a_{ik} = a_{ki} = 0,\ i \neq k; \\ \alpha_i + \alpha_k, & \text{if } a_{ik} \neq 0 \text{ or } a_{ki} \neq 0,\ i \neq k; \end{cases} \qquad \alpha_i \in \Pi. \tag{2}$$

If $\alpha_k \in \Pi$ is non-isotropic, then we define the (even) reflection r_k at α_k by

$$r_k(\alpha) = \beta - \frac{2(\beta,\alpha_k)}{(\alpha_k,\alpha_k)}\alpha_k \qquad \beta \in \Delta. \tag{3}$$

If $\alpha_k \in \Pi$ is an even root, then r_k also satisfies (1). However, if $\alpha_k \in \Pi$ is a non-isotropic odd root, then

$$\Delta'^+ = \big(\Delta^+ \setminus \{\alpha_k, 2\alpha_k\}\big) \cup \{-\alpha_k, -2\alpha_k\}. \tag{4}$$

1.5 The Weyl groupoid

The *Weyl groupoid* $\mathcal{W}$ for a basic Lie superalgebra $\mathfrak{g}$ is a groupoid which acts by even and odd reflections on the set of bases of $\mathfrak{g}$ [18]. For each base $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ and each simple root $\alpha_k \in \Pi$ the set $r_k(\Pi) \subset \Delta$ defined by $r_k(\Pi) = \{r_k(\alpha_n), \ldots, r_k(\alpha_n)\}$ is a base for Δ [18]. The Weyl groupoid acts transitively on the set of bases of a basic Lie superalgebra. Indeed, if we have two different decompositions $\Delta = \Delta^+ \sqcup \Delta^-$ and $\Delta = \Delta''^+ \sqcup \Delta''^-$, then there is a simple root $\alpha_k \in \Delta^+ \setminus \Delta''^+$. Let Δ'^+ be obtained from Δ^+ by the simple reflection r_k. Then by (1) and (4), $|\Delta'^+ \setminus \Delta''^+| < |\Delta^+ \setminus \Delta''^+|$, so the claim follows by induction. If $\Delta^+_{\bar{0}} = \Delta''^+_{\bar{0}}$, then $\Delta^+ \setminus \Delta''^+$ consists entirely of odd roots. So any two Borel subalgebras $\mathfrak{b}, \mathfrak{b}' \subset \mathfrak{g}$ satisfying $\mathfrak{b}_{\bar{0}} = \mathfrak{b}'_{\bar{0}}$ are connected by a chain of odd reflections. In particular, one can use odd reflections to move between the different Dynkin diagrams of a basic Lie superalgebra.

The Weyl groupoid also acts on the set of all "marked bases" of $\mathfrak{g}$. A *marked base* (Π, D) is a base $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ together with an assignment of integers $D = \{d_1, \ldots, d_n\}$. Given a marked base (Π, D), we can extend D linearly to a map $D : \Delta \cup \{0\} \to \mathbb{Z}$ by $D(\beta) = D(\sum_{i=1}^n k_i\alpha_i) = \sum_{i=1}^n k_i d_i$. For each simple root $\alpha_k \in \Pi$, we can reflect at α_k to obtain a marked base $(r_k(\Pi), r_k(D))$, where $r_k(D) = \{D(r_k(\alpha_1)), \ldots, D(r_k(\alpha_n))\}$. For each $i = 1, \ldots, n$, $D(r_k(\alpha_i))$ can be easily computed

from the definition of r_k using linearity. If $D(\alpha_k) = 0$, then $D(r_k(\alpha_i)) = D(\alpha_i)$ for $i = 1, \ldots, n$. Similarly, $\mathcal{W}$ acts on the set of all "marked diagrams" of $\mathfrak{g}$. A *marked diagram* is obtained by assigning an integer to each vertex of a Dynkin diagram of $\mathfrak{g}$.

2 Properties of $\mathbb{Z}$-gradings

A $\mathbb{Z}$-grading of a Lie superalgebra $\mathfrak{g}$ is a decomposition into a direct sum of $\mathbb{Z}/2\mathbb{Z}$-graded subspaces $\mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}(j)$ such that $[\mathfrak{g}(i), \mathfrak{g}(j)] \subset \mathfrak{g}(i+j)$. Let

$$\mathfrak{g}_{\geq} = \oplus_{j\geq 0}\mathfrak{g}(j), \quad \mathfrak{g}_{\leq} = \oplus_{j\leq 0}\mathfrak{g}(j), \quad \mathfrak{g}_{+} = \oplus_{j>0}\mathfrak{g}(j), \quad \text{and} \quad \mathfrak{g}_{-} = \oplus_{j<0}\mathfrak{g}(j).$$

If $\mathfrak{a}$ is a subalgebra of a $\mathbb{Z}$-graded Lie superalgebra $\mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}(j)$, we define $\mathfrak{a}(k) := \mathfrak{a} \cap \mathfrak{g}(k)$. Then $\oplus_{k\in\mathbb{Z}}\mathfrak{a}(k)$ is a subalgebra of $\mathfrak{a}$. We say that that $\mathfrak{a}$ is a *graded subalgebra of* $\mathfrak{g}$ if $\mathfrak{a} = \oplus_{k\in\mathbb{Z}}\mathfrak{a}(k)$, and call this grading the *induced grading*. Clearly, $\mathfrak{g}_{\bar{0}}$ is a graded subalgebra of $\mathfrak{g}$. One can show that the derived subalgebra $\mathfrak{g}'$ and the center $\mathfrak{Z}(\mathfrak{g})$ are also graded subalgebras of $\mathfrak{g}$. Moreover the *centralizer of* T *in* $\mathfrak{g}$, defined by $\mathfrak{g}^T := \{x \in \mathfrak{g} \mid [x,t] = 0 \ \forall t \in T\}$, is a graded subalgebra of $\mathfrak{g}$ when T is spanned by set of homogeneous elements.

2.1 Cartan subalgebras and the root space decomposition

Lemma 2.1. *Let $\mathfrak{g}$ be a basic Lie superalgebra, or let $\mathfrak{g}$ be $\mathfrak{gl}(m|n)$ or $\mathfrak{sl}(n|n)$. Let $\mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}(j)$ be a $\mathbb{Z}$-grading of $\mathfrak{g}$ satisfying $\mathfrak{Z}(\mathfrak{g}_{\bar{0}}) \subset \mathfrak{g}_{\bar{0}}(0)$. Then*

(i) $\mathfrak{g}'_{\bar{0}}(0)$ is a reductive Lie algebra and so is $\mathfrak{g}_{\bar{0}}(0) = \mathfrak{g}'_{\bar{0}}(0) \times \mathfrak{Z}(\mathfrak{g}_{\bar{0}})$;

(ii) there exists a Cartan subalgebra $\mathfrak{h}$ for $\mathfrak{g}$ such that $\mathfrak{h} \subset \mathfrak{g}_{\bar{0}}(0)$.

Proof. Now $\mathfrak{g}_{\bar{0}} = \mathfrak{g}'_{\bar{0}} \times \mathfrak{Z}(\mathfrak{g}_{\bar{0}})$ is reductive, and $\mathfrak{g}_{\bar{0}}(0) = \mathfrak{g}'_{\bar{0}}(0) \times \mathfrak{Z}(\mathfrak{g}_{\bar{0}})$. Since $\mathfrak{g}'_{\bar{0}}$ is a semisimple Lie algebra, there exists $H \in \mathfrak{g}'_{\bar{0}}(0)$ which defines the induced grading of $\mathfrak{g}'_{\bar{0}}$. Now $[H, \mathfrak{Z}(\mathfrak{g}_{\bar{0}})] = 0$ and by assumption $\mathfrak{Z}(\mathfrak{g}_{\bar{0}}) \subset \mathfrak{g}_{\bar{0}}(0)$, hence H defines the induced grading of $\mathfrak{g}_{\bar{0}}$.

It is well known that if $\mathfrak{a} = \oplus_{j\in\mathbb{Z}}\mathfrak{a}(j)$ is a $\mathbb{Z}$-grading of a semisimple Lie algebra and $h \in \mathfrak{a}$ defines the grading, then $\mathfrak{a}(0) = \mathfrak{a}^h$ is a reductive Lie algebra. In particular, $\mathfrak{g}'_{\bar{0}}(0) = (\mathfrak{g}'_{\bar{0}})^H$ is a reductive Lie algebra. Moreover, $\mathfrak{g}_{\bar{0}}(0) = \mathfrak{g}'_{\bar{0}}(0) \times \mathfrak{Z}(\mathfrak{g}_{\bar{0}})$ is a reductive Lie algebra.

Since H is semisimple in $\mathfrak{g}_{\bar{0}}$ we can choose a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}_{\bar{0}}(0)$ which contains H. Then $\mathfrak{h}$ is a Cartan subalgebra for $\mathfrak{g}_{\bar{0}}$, and so by [9] $\mathfrak{h}$ is a Cartan subalgebra for $\mathfrak{g}$. ☐

Remark 2.2. If $\mathfrak{g}$ is a basic Lie superalgebra with a $\mathbb{Z}$-grading $\mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}(j)$, then $\mathfrak{Z}(\mathfrak{g}_{\bar{0}}) \subset \mathfrak{g}(0)$. Indeed, elements of $\mathfrak{Z}(\mathfrak{g}_{\bar{0}})$ are ad-semisimple on $\mathfrak{g}$ since the action of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ is completely reducible, whereas elements of $\mathfrak{g}(j)$ for $j \neq 0$ are ad-nilpotent on $\mathfrak{g}$. The claim follows since $\mathfrak{Z}(\mathfrak{g}) = 0$.

Lemma 2.3. *Let $\mathfrak{g}$ be a basic Lie superalgebra, $\mathfrak{g} \neq \mathfrak{psl}(2|2)$, or let $\mathfrak{g}$ be $\mathfrak{gl}(m|n)$ or $\mathfrak{sl}(n|n)$. Fix a $\mathbb{Z}$-grading $\mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}(j)$ satisfying $\mathfrak{Z}(\mathfrak{g}_{\bar{0}}) \subset \mathfrak{g}_{\bar{0}}(0)$. Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}_{\bar{0}}(0)$. Let $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta}\mathfrak{g}_\alpha$ be the corresponding root space decomposition of $\mathfrak{g}$.*

(i) For each $\alpha \in \Delta$, there exist $k \in \mathbb{Z}$ such that $\mathfrak{g}_\alpha \subset \mathfrak{g}(k)$. Thus

$$\mathfrak{g}(0) = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta_0}\mathfrak{g}_\alpha \quad \textit{and} \quad \mathfrak{g}(j) = \bigoplus_{\alpha\in\Delta_j}\mathfrak{g}_\alpha \quad \textit{for each} \quad j \in \mathbb{Z}, \ j \neq 0,$$

where $\Delta_j = \{\alpha \in \Delta \mid \mathfrak{g}_\alpha \subset \mathfrak{g}_j\}$.

(ii) Define the degree map Deg : $\Delta \cup \{0\} \to \mathbb{Z}$ *by* $Deg(\alpha) = k$ *if* $\alpha \in \Delta_k$ *and* $Deg(0) = 0$. *Then Deg is a linear map,* $Deg(-\alpha) = -Deg(\alpha)$ *for all* $\alpha \in \Delta$, *and* $\Delta_{-j} = -\Delta_j$ *for all* $j \in \mathbb{Z}$.

Proof. Fix $\alpha \in \Delta$ and let $x \in \mathfrak{g}_\alpha$, $x \neq 0$. Write $x = \sum_{j\in\mathbb{Z}} x_j$ where $x_j \in \mathfrak{g}(j)$. Then for each $h \in \mathfrak{h}$ we have that

$$\sum_{j\in\mathbb{Z}}[h, x_j] = [h, x] = \alpha(h)x = \sum_{j\in\mathbb{Z}} \alpha(h)x_j$$

Since $\mathfrak{h}$ preserves each graded component, this implies $[h, x_j] = \alpha(h)x_j$ for each $h \in \mathfrak{h}$ and $j \in \mathbb{Z}$. Thus $x_j \in \mathfrak{g}_\alpha$ for each $j \in \mathbb{Z}$. For $\mathfrak{g} \neq \mathfrak{psl}(2|2)$, $\dim \mathfrak{g}_\alpha = 1$ implies that $x = x_k \in \mathfrak{g}(k)$ for some $k \in \mathbb{Z}$. Hence, $\mathfrak{g}_\alpha = \mathbb{C}x \subset \mathfrak{g}(k)$ for some $k \in \mathbb{Z}$. Now $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$, and by Theorem 1.1, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \neq 0$ when $\alpha, \beta, \alpha+\beta \in \Delta\cup\{0\}$. Thus, $\mathrm{Deg}(\alpha)+\mathrm{Deg}(\beta) = \mathrm{Deg}(\alpha+\beta)$ for $\alpha, \beta, \alpha+\beta \in \Delta\cup\{0\}$. □

2.2 Inner derivations

Let $\mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}(j)$ be a $\mathbb{Z}$-grading of a Lie superalgebra $\mathfrak{g}$. The linear map $\phi : \mathfrak{g} \to \mathfrak{g}$ defined by $\sum_{j\in\mathbb{Z}} x_j \mapsto \sum_{j\in\mathbb{Z}} jx_j$, with $x_j \in \mathfrak{g}(j)$, is a derivation. If $\mathfrak{g}$ is a semisimple Lie algebra or a basic Lie superalgebra, $\mathfrak{g} \neq \mathfrak{psl}(n|n)$, then all derivations of $\mathfrak{g}$ are inner [9, 17]. So there exists $H \in \mathfrak{g}$ that defines the grading, that is, $[H, x_j] = jx_j$ for all $x_j \in \mathfrak{g}(j)$. Since ϕ preserves parity, $H \in \mathfrak{g}_{\bar{0}}$.

Lemma 2.4. *A* $\mathbb{Z}$*-grading of* $\mathfrak{g} = \mathfrak{sl}(n|n)$ *or* $\mathfrak{gl}(n|n)$ *which satisfies* $\mathfrak{z}(\mathfrak{g}_{\bar{0}}) \subset \mathfrak{g}(0)$ *is defined by an inner derivation of* $\mathfrak{gl}(n|n)$.

Proof. We can extend a $\mathbb{Z}$-grading of $\mathfrak{sl}(n|n)$ to a $\mathbb{Z}$-grading of $\mathfrak{g} = \mathfrak{gl}(n|n)$ such that $\mathfrak{z}(\mathfrak{g}_{\bar{0}}) \subset \mathfrak{g}(0)$. Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}(0)$. By Lemma 2.3, the $\mathbb{Z}$-grading is compatible with the root space decomposition. A $\mathbb{Z}$-grading is determined the value of the degree map on a set of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_{2n-1}\} \subset \Delta$. Since Π is a linearly independent set in $\mathfrak{h}^*$, there exists $H \in \mathfrak{h}$ such that $\alpha_i(H) = \mathrm{Deg}(\alpha_i)$ for $i = 1, \ldots, 2n-1$. Clearly, ad H defines the $\mathbb{Z}$-grading of $\mathfrak{sl}(n|n) \subset \mathfrak{gl}(n|n)$. □

2.3 $\mathbb{Z}$-gradings of $\mathfrak{psl}(n|n)$

Lemma 2.5. *For* $n \neq 2$, *a* $\mathbb{Z}$*-grading of* $\mathfrak{psl}(n|n)$ *is induced from a* $\mathbb{Z}$*-grading of* $\mathfrak{sl}(n|n)$, *which satisfies* $\mathfrak{z}(\mathfrak{sl}(n|n)_{\bar{0}}) \subset \mathfrak{sl}(n|n)(0)$.

Proof. Fix a $\mathbb{Z}$-grading of $\mathfrak{psl}(n|n)$, and a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}(0)$. Let $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ be a set of simple roots. By Lemma 2.3, $e_i \in \mathfrak{g}_{\alpha_i}$ and $f_i \in \mathfrak{g}_{-\alpha_i}$ are homogeneous for $i = 1, \ldots, n$, and $\deg(-\alpha_i) = -\deg(\alpha_i)$. Let $d_i = \mathrm{Deg}(\alpha_i)$ for $i = 1, \ldots, n$. This determines uniquely a $\mathbb{Z}$-grading of $\mathfrak{sl}(n|n)$ with $\mathfrak{z}(\mathfrak{sl}(n|n)) \subset \mathfrak{sl}(n|n)(0)$. Since $\mathfrak{psl}(n|n)$ is generated by $e_1, \ldots, e_n, f_1, \ldots, f_n$, the quotient of this $\mathbb{Z}$-grading of $\mathfrak{sl}(n|n)$ by the center must coincide with the original $\mathbb{Z}$-grading of $\mathfrak{psl}(n|n)$. In particular, $\mathfrak{psl}(n|n)(j) = \mathfrak{sl}(n|n)(j)$ for all $j \in \mathbb{Z}\setminus\{0\}$ and $\mathfrak{psl}(n|n)(0) = \mathfrak{sl}(n|n)(0)/\mathfrak{z}(\mathfrak{sl}(n|n))$. □

The following lemma can be proved by explicit computation.

Lemma 2.6. *The* $\mathbb{Z}$*-gradings of* $\mathfrak{g} = \mathfrak{psl}(2|2)$ *(up to conjugation by* $\mathfrak{g}_{\bar{0}}$*) are parameterized as follows. For each* $m \in \mathbb{Z}$, $p, q \in \{0, 2\}$, *and* $(a : b), (c : d) \in \mathbb{P}^2$ *satisfying* $(a : b) \neq (c : d)$ *we obtain a* $\mathbb{Z}$*-grading from the following assignment of degrees to the linear basis (of representatives modulo the*

center $\mathbb{C}(I_4)$*):*

$$\begin{array}{lll}
Deg(E_{12}) = p & Deg(aE_{14} + bE_{32}) = m & Deg(bE_{31} - aE_{24}) = m - p \\
Deg(E_{34}) = q & Deg(dE_{41} + cE_{23}) = -m & Deg(cE_{13} - dE_{42}) = p - m \\
Deg(E_{21}) = -p & Deg(cE_{14} + dE_{32}) = p + q - m & Deg(dE_{31} - cE_{24}) = q - m \\
Deg(E_{43}) = -q & Deg(bE_{41} + aE_{23}) = m - p - q & Deg(aE_{13} - bE_{42}) = m - q \\
Deg\ (E_{11} + E_{33}) = 0 & Deg\ (E_{22} + E_{44}) = 0 &
\end{array}$$

2.4 The bilinear form

Lemma 2.7. *Let $\mathfrak{g}$ be a basic Lie superalgebra, or let $\mathfrak{g}$ be $\mathfrak{gl}(m|n)$ or $\mathfrak{sl}(n|n)$ with $(\cdot,\cdot)$ as defined in Section 1.3. Fix a $\mathbb{Z}$-grading $\mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}(j)$ satisfying $\mathfrak{Z}(\mathfrak{g}) \subset \mathfrak{g}(0)$. Then*

(i) $(\mathfrak{g}(i), \mathfrak{g}(j)) = 0$ *for* $i \neq -j$*;*

(ii) $(x, \mathfrak{g}(j)) \neq 0$ *for* $x \in \mathfrak{g}_{-j}$*,* $x \notin \mathfrak{Z}(\mathfrak{g})$*.*

Proof. This follows from Theorem 1.1, Lemma 2.3 and Lemma 2.6. □

2.5 Characteristics and the action of the Weyl groupoid

We can represent a $\mathbb{Z}$-grading by the values of the degree map on a set of simple roots. In this section, we examine the properties of the degree map with respect to different sets of simple roots.

Let $\mathfrak{g}$ be a basic Lie superalgebra, $\mathfrak{g} \neq \mathfrak{psl}(2|2)$, or let $\mathfrak{g}$ be $\mathfrak{gl}(m|n)$ or $\mathfrak{sl}(n|n)$. Fix a $\mathbb{Z}$-grading $\mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}(j)$ satisfying $\mathfrak{Z}(\mathfrak{g}_{\bar{0}}) \subset \mathfrak{g}_{\bar{0}}(0)$. Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}_{\bar{0}}(0)$, and let Δ be the set of roots. Let $\Delta_{\geq} = \{\alpha \in \Delta | \mathrm{Deg}(\alpha) \geq 0\}$ and $\Delta_{<} = \{\alpha \in \Delta | \mathrm{Deg}(\alpha) < 0\}$.

Lemma 2.8. *There exists a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}_{\geq}$, $\Delta^+ \subset \Delta_{\geq}$. In particular, $\mathfrak{g}_{\geq}$ is a parabolic subalgebra with nilradical $\mathfrak{g}_+$ and Levi subalgebra $\mathfrak{g}(0)$.*

Proof. Fix a $\mathbb{Z}$-grading $\mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}(j)$ with $\mathfrak{Z}(\mathfrak{g}_{\bar{0}}) \subset \mathfrak{g}(0)$. Let Π be a base of Δ and $\Delta = \Delta^+ \sqcup \Delta^-$. If $\Delta^+ \not\subset \Delta_{\geq}$, then there is $\alpha_k \in \Pi \cap \Delta_{<}$. Let r_k denote the reflection of Δ with respect to α_k (see Section 1.4). Then $\Pi' := \{r_k(\alpha_1), \ldots, r_k(\alpha_n)\}$ is a base for $\mathfrak{g}$ with decomposition $\Delta' = \Delta'^+ \sqcup \Delta'^-$. By (1), (4) and Lemma 2.3, $|\Delta'^+ \cap \Delta_<| = |\Delta^+ \cap \Delta_<| - |\{\alpha_k, 2\alpha_k\}| < |\Delta^+ \cap \Delta_<|$. Since Δ is finite, the claim follows by induction. □

It may be possible to choose more than one Borel subalgebra in $\mathfrak{g}_{\geq}$. However, we have

Lemma 2.9. *Suppose $\Delta_1^+, \Delta_2^+ \subset \Delta_{\geq}$. Let $\gamma \in \Delta_1^+ \setminus \Delta_2^+$. Then $Deg(\gamma) = -Deg(-\gamma) = 0$.*

Proof. By Lemma 2.3, $-\gamma \in \Delta_2^+$. So $\pm\gamma \in \Delta_{\geq}$ implying $\mathrm{Deg}(\gamma) = -\mathrm{Deg}(-\gamma) = 0$. □

Now for each base $\Pi \subset \Delta$, the degree map of a $\mathbb{Z}$-grading is determined by its restriction to Π, that is, by $D : \Pi \to \mathbb{Z}$. A reflection at a simple root of Π yields a new map $D' : \Pi' \to \mathbb{Z}$, where Π' is the reflected base and D' is defined on Π' by linearity (see Section 1.5). The maps $D : \Pi \to \mathbb{Z}$ and $D' : \Pi' \to \mathbb{Z}$ define the same grading on Δ. Moreover, any map $D' : \Pi' \to \mathbb{Z}$ obtained from $D : \Pi \to \mathbb{Z}$ by the action of the Weyl groupoid $\mathcal{W}$ defines the same grading on Δ.

If $\Pi \subset \Delta_+$, then the induced map $\mathrm{Deg} : \Pi \to \mathbb{N}$ is called the *characteristic* of the $\mathbb{Z}$-grading with respect to Π. It is natural to ask the following question: when do two maps $D_1 : \Pi_1 \to \mathbb{N}$ and $D_2 : \Pi_2 \to \mathbb{N}$ define the same $\mathbb{Z}$-grading, i.e. when can they be extended to a linear map $\mathrm{Deg} : \Delta \cup \{0\} \to \mathbb{Z}$?

Theorem 2.10. *Let $\Pi_1 = \{\alpha_1, \ldots, \alpha_n\}$, $\Pi_2 = \{\beta_1, \ldots, \beta_n\}$ be two different bases for Δ. If the maps $D_1 : \Pi_1 \to \mathbb{N}$ and $D_2 : \Pi_2 \to \mathbb{N}$ define the same grading, then there is a sequence of even and odd reflections $\mathcal{R}$ at simple roots of degree zero such that (after reordering) $\mathcal{R}(\alpha_i) = \beta_i$ and $D_1(\alpha_i) = D_2(\beta_i)$ for $i = 1, \ldots, n$.*

Proof. Suppose that $\mathrm{Deg} : \Delta \cup \{0\} \to \mathbb{Z}$ is a linear map whose restriction to Π_1 is D_1 and to Π_2 is D_2. Let $\alpha_k \in \Pi_1$ such that $\alpha_k \notin \Delta_2^+$, and let r_k be the reflection at α_k. By Lemma 2.9, $\mathrm{Deg}(\alpha_k) = 0$ which implies $\mathrm{Deg}(r_k(\alpha_i)) = \mathrm{Deg}(\alpha_i)$ for $i = 1, \ldots, n$. Let $\Pi' := \{r_k(\alpha_1), \ldots, r_k(\alpha_n)\}$ and let $\Delta' = \Delta'^+ \sqcup \Delta'^-$ be the corresponding decomposition. Then $|\Delta'^+ \setminus \Delta_2^+| < |\Delta_1^+ \setminus \Delta_2^+|$. Since Δ is finite, it follows by induction that there is a sequence of reflections $\mathcal{R}$ at simple roots of degree zero such that $R(\alpha_i) = \beta_i$ and $D_1(\alpha_i) = \mathrm{Deg}(\alpha_i) = \mathrm{Deg}(\beta_i) = D_2(\beta_i)$ for $i = 1, \ldots, n$. $\square$

Given a map $D : \Pi \to \mathbb{N}$, an even reflection at a simple root of positive degree yields a map $D' : \Pi' \to \mathbb{Z}$ with $\mathrm{Im}\, D' \not\subset \mathbb{N}$. Whereas, an even reflection at a simple root of degree zero does not change D or the Dynkin diagram corresponding to Π. Thus, a $\mathbb{Z}$-grading is determined up to equivalence by a Dynkin diagram Γ and a labeling the vertices of Γ by nonnegative integers. Hence, for a simple Lie algebra $\mathfrak{g}$, there is a bijection between all $\mathbb{Z}$-gradings of $\mathfrak{g}$ up to conjugation and all characteristics of the Dynkin diagram [6]. However, a Lie superalgebra usually has more than one Dynkin diagram.

Two Dynkin diagrams Γ_1, Γ_2 for a basic Lie superalgebra $\mathfrak{g}$ with degree maps $D_i : \Gamma_i \to \mathbb{N}$ define the same $\mathbb{Z}$-grading if and only if there is a sequence of odd reflections $\mathcal{R}$ at simple isotropic roots of degree zero such that $\mathcal{R}(\Gamma_1) = \Gamma_2$ and $D_1 = D_2$ with the ordering of the vertices defined by $\mathcal{R}$. This defines an equivalence relation on Dynkin diagrams with nonnegative integer labels. We note that if a marked diagram has no isotropic roots of degree zero, then there is only one member in its equivalence class.

3 Centralizers in basic Lie superalgebras

3.1 Nilpotent even elements

Let $\mathfrak{g}$ be a basic Lie superalgebra, or let $\mathfrak{g}$ be $\mathfrak{gl}(m|n)$ or $\mathfrak{sl}(n|n)$. In this section we discuss the orbits of nilpotent even elements in $\mathfrak{g}$ under the action of the group of inner automorphisms G. Recall that G is the group of automorphisms of $\mathfrak{g}$ generated by $\exp(\mathrm{ad}\, x)$ for $x \in \mathfrak{g}_{\bar{0}}$. For $x \in \mathfrak{g}$, let Gx denote the *orbit* of x in $\mathfrak{g}$ under the action of G. An element $e \in \mathfrak{g}$ is called *nilpotent* if the action of ad e on $\mathfrak{g}$ is nilpotent.

Let $e \in \mathfrak{g}$ be a nilpotent even element. Then $e \in \mathfrak{g}'_{\bar{0}}$, since elements of $\mathfrak{Z}(\mathfrak{g}_{\bar{0}})$ are semisimple in $\mathfrak{g}$ [9]. So $Ge \subset \mathfrak{g}'_{\bar{0}}$. Let G' be the group of automorphisms of $\mathfrak{g}'_{\bar{0}}$ generated by $\exp(\mathrm{ad}\, e)$ for $e \in \mathfrak{g}'_{\bar{0}}$. It follows that $Ge = G'e \subset \mathfrak{g}'_{\bar{0}}$. Moreover, if $e \in \mathfrak{g}'_{\bar{0}}$ is ad-nilpotent on $\mathfrak{g}'_{\bar{0}}$, then e is ad-nilpotent on $\mathfrak{g}$. This follows from the Jacobson-Morosov Theorem and $\mathfrak{sl}_2$ theory since $\mathfrak{g}$ is finite dimensional. Thus we are reduced to studying nilpotent orbits in the semisimple Lie algebra $\mathfrak{g}'_{\bar{0}}$.

Let $m, n \in \mathbb{Z}_+$. We say that (p, q) is a partition of $(m|n)$ if p is a partition of m and q is a partition of n. There is a one-to-one correspondence between G-orbits of nilpotent even elements in $\mathfrak{gl}(m|n)$ and partitions of $(m|n)$. Two nilpotent even elements of $\mathfrak{osp}(m|2n)$ belong to the same $O(m) \times SP(2n)$ orbit if and only if they belong the same $GL(m) \times GL(2n)$ orbit. This follows from the theory of nilpotent orbits in finite-dimensional simple Lie algebras (see for example [8]).

Given a partition p, we let $p_1 > \cdots > p_a$ be the distinct nonzero parts of p, and we write $p = (p_1^{m_{p_1}}, \ldots, p_a^{m_{p_a}})$, where m_{p_i} is the multiplicity of p_i in p. A partition is called *symplectic*

(resp. *orthogonal*) if m_{p_i} is even for odd p_i (resp. even p_i). We say that a partition $(p|q)$ of $(m|2n)$ is orthosymplectic if p is an orthogonal partition of m and q is a symplectic partition of $2n$. There is a one-to-one correspondence between G-orbits of nilpotent even elements in $\mathfrak{osp}(m|2n)$ and orthosymplectic partitions of $(m|n)$. See Section 8 for a description of orbit representatives.

3.2 Centralizers of nilpotent even elements

In this section, we describe the centralizer $\mathfrak{g}^e$ of a nilpotent even element $e \in \mathfrak{g}$ by choosing a nice basis of $V_0 \oplus V_1$ (and hence of $\mathrm{End}(V_0 \oplus V_1)$), which we use to compute the dimensions of $\mathfrak{g}^e_{\bar{0}}$ and $\mathfrak{g}^e_{\bar{1}}$. This is analogous to the Lie algebra case [8]. This was done for $\mathfrak{gl}(m|n)$ in [19] for a field of prime characteristic, but the construction is identical in characteristic zero.

3.2.1 Centralizers of nilpotent even elements in $\mathfrak{gl}(m|n)$

Let $\mathfrak{g} = \mathfrak{gl}(m|n) := \mathrm{End}(V_0 \oplus V_1)$, where $\mathfrak{g}_{\bar{0}} = \mathrm{End}(V_0) \oplus \mathrm{End}(V_1)$, $\mathfrak{g}_{\bar{1}} = \mathrm{Hom}(V_0, V_1) \oplus \mathrm{Hom}(V_1, V_0)$ and $\dim V_0 = m$, $\dim V_1 = n$. Let $e \in \mathfrak{g}$ be a nilpotent element such that $e \in \mathfrak{g}_{\bar{0}}$. Then e corresponds to some partition (p, q) of $(m|n)$ given by positive integers $p_1 \geq \cdots \geq p_r$, $q_1 \geq \cdots \geq q_s$, respectively.

Since e is a nilpotent element in $\mathrm{End}(V_0) \oplus \mathrm{End}(V_1)$, there exist $v_1, \ldots, v_r \in V_0$, $w_1, \ldots, w_s \in V_1$ such that $\{e^j v_i \mid 1 \leq i \leq r,\ 0 \leq j < p_i\}$ is a basis for V_0 and $\{e^j w_i \mid 1 \leq i \leq s,\ 0 \leq j < q_i\}$ is a basis for V_1, and $e^{p_i} v_i = 0$, $e^{q_i} w_i = 0$ by [8]. Each element $Z \in \mathfrak{g}^e$ is determined by $Z(v_i), 1 \leq i \leq r$ and $Z(w_i), 1 \leq i \leq s$, since $Z(e^j v_i) = e^j Z(v_i)$ and $Z(e^j w_i) = e^j Z(w_i)$.

For $Z \in \mathfrak{g}^e_{\bar{0}}$ one has

$$Z(v_i) = \sum_{j=1}^{r} \sum_{k=max\{0, p_j - p_i\}}^{p_j - 1} \alpha_{k,j:i} e^k v_j, \qquad Z(w_i) = \sum_{j=1}^{s} \sum_{k=max\{0, q_j - q_i\}}^{q_j - 1} \beta_{k,j:i} e^k w_j \tag{5}$$

For $Z \in \mathfrak{g}^x_{\bar{1}}$ one has

$$Z(v_i) = \sum_{j=1}^{s} \sum_{k=max\{0, q_j - p_i\}}^{q_j - 1} \gamma_{k,j:i} e^k w_j, \qquad Z(w_i) = \sum_{j=1}^{r} \sum_{k=max\{0, p_i - q_j\}}^{p_i - 1} \delta_{k,j:i} e^k v_j \tag{6}$$

Since the coefficients $\alpha_{k,j:i}, \beta_{k,j:i}, \gamma_{k,j:i}, \delta_{k,j:i}$ can be chosen arbitrarily, the dimensions of $\mathfrak{g}^e_{\bar{0}}$ and $\mathfrak{g}^e_{\bar{1}}$ are determined by the number of indices. Hence by [8, 19], we have

$$\begin{aligned}
\dim\, \mathfrak{g}^e_{\bar{0}} &= \sum_{i,j=1}^{r} \min(p_i, p_j) + \sum_{i,j=1}^{s} \min(q_i, q_j) \\
&= \left(m + 2\sum_{i=1}^{r} (i-1) p_i \right) + \left(n + 2\sum_{j=1}^{s} (j-1) q_j \right), \\
\dim\, \mathfrak{g}^e_{\bar{1}} &= 2 \sum_{i,j=1}^{r,s} \min(p_i, q_j).
\end{aligned}$$

The following lemma will be used in the proof of Theorem 7.2.

Lemma 3.1. *Let $\mathfrak{a} = \mathfrak{gl}(m) \times \mathfrak{gl}(n)$ and let $i : \mathfrak{a} \hookrightarrow \mathfrak{gl}(m|n)$, $j : \mathfrak{a} \hookrightarrow \mathfrak{gl}(m+n)$ be the natural inclusion maps. Then $\mathfrak{gl}(m|n)$ and $\mathfrak{gl}(m+n)$ are isomorphic as $\mathfrak{a}$-modules under the adjoint action. Hence dim $\mathfrak{gl}(m|n)^{i(x)} = dim\ \mathfrak{gl}(m+n)^{j(x)}$ for all $x \in \mathfrak{a}$.*

For $m \in \mathbb{N}$, let $Par(m)$ denote the set of partitions of m. Then for $m, n \in \mathbb{N}$ we have a natural map $\psi_{m,n} : Par(m) \times Par(n) \to Par(m+n)$.

Lemma 3.2. *If e is a nilpotent element corresponding to the partition (p,q) of $(m|n)$ and $\psi_{m,n}(p,q) = r = (r_1, \dots, r_k) \in Par(m+n)$, then dim $\mathfrak{gl}(m|n)^e = \mathfrak{gl}(m+n)^e = (r_1^*)^2 + \cdots (r_N^*)^2$, where $(r_1^*, \dots, r_N^*)$ is the dual partition.*

3.2.2 Centralizers of nilpotent even elements in $\mathfrak{osp}(m|2n)$

Now $\mathfrak{g} = \mathfrak{osp}(m|2n) \subset \mathfrak{gl}(m|2n) = \mathrm{End}(V_0 \oplus V_1)$ is defined as follows. Let φ be a non-degenerate supersymmetric bilinear form on $V = V_0 \oplus V_1$, so that V_0 and V_1 are orthogonal and the restriction to V_0 is symmetric while the restriction to V_1 is skew-symmetric. Then $\varphi(x,y) = (-1)^{\deg x \ \deg y}\varphi(y,x)$ for all homogeneous elements $x, y \in V$. Define

$$\mathfrak{osp}(m|2n)_i := \{z \in \mathfrak{gl}(m|2n)_i \mid \varphi(z(x), y) = -(-1)^{i(\deg\ x)}\varphi(x, z(y))\}.$$

Let $e \in \mathfrak{g}$ be a nilpotent element in $\mathfrak{g}$ such that $e \in \mathfrak{g}_{\bar{0}}$. Then e corresponds to some orthosymplectic partition (p,q) of $(m|n)$ given by $p_1 \geq p_2 \geq \cdots \geq p_r > 0$, $q_1 \geq q_2 \geq \cdots \geq q_s > 0$. Since $e \in \mathrm{End}(V_0) \oplus \mathrm{End}(V_1)$, by [8] there exist $v_1, \dots, v_r \in V_0$, $w_1, \dots, w_s \in V_1$ such that $\{e^j v_i \mid 1 \leq i \leq r,\ 0 \leq j < p_i\}$ is a basis for V_0 and $\{e^j w_i \mid 1 \leq i \leq s,\ 0 \leq j < q_i\}$ is a basis for V_1, $e^{p_i} v_i = 0$, $e^{q_i} w_i = 0$, and which satisfy the following:

- for each odd p_i,

$$\varphi(e^j v_i, e^h v_k) = \begin{cases} (-1)^j, & \text{if } k = i \text{ and } j + h = p_i - 1; \\ 0, & \text{otherwise;} \end{cases}$$

- for each even p_i, there exists $\mu_i \in \{0,1\}$ and index $i^* \neq i$, $1 \leq i^* \leq r$, such that $p_{i^*} = p_i$ and

$$\varphi(e^j v_i, e^h v_k) = \begin{cases} (-1)^j \mu_i, & \text{if } k = i^* \text{ and } j + h = p_i - 1; \\ 0, & \text{otherwise;} \end{cases}$$

- for each odd q_i, there exists $\omega_i \in \{0,1\}$ and index $i^* \neq i$, $1 \leq i^* \leq s$, such that $q_{i^*} = q_i$ and

$$\varphi(e^j w_i, e^h w_k) = \begin{cases} (-1)^j \omega_i, & \text{if } k = i^* \text{ and } j + h = q_i - 1; \\ 0, & \text{otherwise;} \end{cases}$$

- for each even q_i

$$\varphi(e^j w_i, e^h w_k) = \begin{cases} (-1)^j, & \text{if } k = i \text{ and } j + h = q_i - 1; \\ 0, & \text{otherwise.} \end{cases}$$

Note that $\varphi(e^j v_i, e^h w_k) = 0$ for all h, i, j, k.

Now $\mathfrak{g}^e = \mathfrak{g} \cap \mathfrak{gl}(m|2n)^e$, so let $Z \in \mathfrak{gl}(m|2n)^e$. For $Z \in \mathfrak{g}^e_{\bar{0}} = \mathfrak{so}(m)^e \times \mathfrak{sp}(2n)^e$, the coefficients $\alpha_{k,j:i}, \beta_{k,j:i}$ of $Z(v_i)$ and $Z(w_i)$ in (5) satisfy certain conditions given in [8, Section 3.2], and since $\dim\, \mathfrak{g}^e_{\bar{0}} = \dim(\mathfrak{so}(m)^e) + \dim(\mathfrak{sp}(2n)^e)$, we have that $\dim\, \mathfrak{g}^e_{\bar{0}} =$

$$\left(\frac{m}{2} + \sum_{i=1}^{r}(i-1)p_i - \frac{1}{2}|\{i \mid p_i \text{ odd}\}|\right) + \left(\frac{n}{2} + \sum_{j=1}^{s}(j-1)q_j + \frac{1}{2}|\{j \mid q_j \text{ odd}\}|\right).$$

For $Z \in \mathfrak{g}^e_{\bar{1}}$, the coefficients $\gamma_{k,j:i}$ of $Z(v_i)$ in (6) can be chosen freely, but then the coefficients $\delta_{k,j:i}$ of $Z(w_i)$ are completely determined from this choice. So the dimension of $\mathfrak{osp}(m|2n)^e_{\bar{1}}$ is one-half the dimension of $\mathfrak{gl}(m|2n)^e_{\bar{1}}$. Hence,

$$\dim\, \mathfrak{g}^e_{\bar{1}} = \sum_{i,j=1}^{r,s} \min(p_i, q_j).$$

3.3 Centralizers of $\mathfrak{sl}_2$-triples

Fix an $\mathfrak{sl}_2$-triple $\mathfrak{s} = \{e, f, h\} \subset \mathfrak{g}'_{\bar{0}}$ satisfying $[e, f] = h$, $[h, e] = 2e$ and $[h, f] = -2f$. It is uniquely determined up to conjugacy by the nilpotent element e [12]. Let $\mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}(j)$ be the $\mathbb{Z}$-grading given by the eigenspaces of ad h. Then $\mathfrak{g}^e = \oplus_{j\geq 0}\mathfrak{g}^e(j)$ and $\mathfrak{g}^{\mathfrak{s}} = \mathfrak{g}^e(0)$. The following lemma can be proven using the same argument as in [8].

Lemma 3.3. *$\mathfrak{g}^e$ is the semidirect product of the subalgebra $\mathfrak{g}^e(0) = \mathfrak{g}^{\mathfrak{s}}$ and the ideal $\oplus_{m>0}\mathfrak{g}^e(m)$. This ideal consists of nilpotent elements.*

Let $(p|q)$ be a partition of $(m|n)$. Let $r \in Par(m + n)$ be the total ordering of the partitions p and q. Let $r_1 > \cdots > r_N$ be the set of distinct nonzero parts of r. Write $r = (r_1^{m_1+n_1}, \ldots, r_N^{m_N+n_N})$ where $p = (r_1^{m_1}, \ldots, r_N^{m_N})$ and $q = (r_1^{n_1}, \ldots, r_N^{n_N})$. Note that for each i, one of m_i, n_i can be zero.

For each $t \in \mathbb{Z}_+$, let $M_t = \sum_{i:p_i=t}\mathbb{F}v_i + \sum_{i:q_i=t}\mathbb{F}w_i$, where v_i, w_i are as given in Section 3.2.1.

Theorem 3.4. *Let $\mathfrak{g} = \mathfrak{gl}(m|n)$. Let e be a nilpotent even element corresponding to a partition (p, q) of $(m|n)$, and let $\mathfrak{s} = \{e, f, h\} \subset \mathfrak{g}'_{\bar{0}}$ be an $\mathfrak{sl}_2$-triple for e. Then we have an isomorphism $\mathfrak{g}^{\mathfrak{s}} \xrightarrow{\sim} \mathfrak{gl}(m_1, n_1) \times \cdots \times \mathfrak{gl}(m_N, n_N)$ of Lie superalgebras.*

Proof. If $Z \in \mathfrak{g}^{\mathfrak{s}} \subset \mathfrak{g}^e$, then a coefficient of $Z(v_i)$, $Z(w_i)$ in (5), (6) is zero unless $k = 0$ and $p_i, q_i = p_j, q_j$. So $Z(M_t) \subset M_t$. Since Z is determined by all the $Z(v_i)$, $Z(w_i)$, the map

$$\mathfrak{g}^e(0) \to \mathfrak{gl}(M_1) \times \mathfrak{gl}(M_2) \times \mathfrak{gl}(M_3) \times \cdots \tag{7}$$

defined by restriction is injective. Since these coefficients can be chosen freely, this map is surjective. The even (resp. odd) dimension of M_t is the number of p_i (resp. q_i) with $p_i = t$ (resp. $q_i = t$). $\square$

Let $(p|q)$ be an orthosymplectic partition of $(m|2n)$. Let r (resp. s) be the total ordering of the odd parts (resp. even parts) of the partitions p and q. Write $r = (r_1^{m_1+2n_1}, \ldots, r_N^{m_N+2n_N})$ and $s = (s_1^{2c_1+d_1}, \ldots, s_T^{2c_T+d_T})$ where $p = (r_1^{m_1}, \ldots, r_N^{m_N}, s_1^{2c_1}, \ldots, s_T^{2c_T})$ and $q = (r_1^{2n_1}, \ldots, r_N^{2n_N}, s_1^{d_1}, \ldots, s_T^{d_T})$.

Theorem 3.5. *Let $\mathfrak{g} = \mathfrak{osp}(m|2n)$. Let e be a nilpotent even element corresponding to an orthosymplectic partition (p, q) of $(m|n)$, and let $\mathfrak{s} = \{e, f, h\} \subset \mathfrak{g}'_{\bar{0}}$ be an $\mathfrak{sl}_2$-triple for e. Then we have an isomorphism*

$$\mathfrak{g}^{\mathfrak{s}} \xrightarrow{\sim} \mathfrak{osp}(m_1, 2n_1) \times \cdots \times \mathfrak{osp}(m_N, 2n_N) \times \mathfrak{osp}(d_1, 2c_1) \times \cdots \times \mathfrak{osp}(d_T, 2c_T)$$

of Lie superalgebras.

Proof. For each $t \in \mathbb{Z}_+$ define a bilinear form on M_t by $\psi_t(x, y) = \varphi(x, e^{t-1}y)$. Then ψ_t is nondegenerate since for $v_i, v_k, w_i, w_k \in M_t$, we have $\psi_t(v_i, v_k) = \delta_{k,i^*}\mu_i$, $\psi_t(w_i, w_k) = \delta_{k,i^*}\omega_i$. Moreover, ψ_t is supersymmetric if t is odd, and skew-supersymmetric if t is even. Indeed, for homogeneous elements $x, y \in M_t$ we have

$$\begin{aligned}\psi_t(x, y) &= \varphi(x, e^{t-1}y) = (-1)^{(\deg\ x)(\deg\ y)}\varphi(e^{t-1}y, x)\\ &= (-1)^{(\deg\ x)(\deg\ y)+(t-1)}\varphi(y, e^{t-1}x)\\ &= (-1)^{(\deg\ x)(\deg\ y)+(t-1)}\psi_t(y, x).\end{aligned}$$

It is clear that $(M_t)_0$ is orthogonal to $(M_t)_1$ with respect to ψ_t for all $t \in \mathbb{Z}_+$, since $e \in \mathfrak{g}_{\bar{0}}$. Let $\Pi(N)$ be the superspace isomorphic to N with switched parity. Then for each $t \in 2\mathbb{Z}_+$, the bilinear form $\psi_t : \Pi(M_t) \times \Pi(M_t) \to \mathbb{C}$ is supersymmetric.

If $Z \in \mathfrak{g}^e(0)_i$, then $Z(M_t) \subset M_t$, and for homogeneous $x, y \in M_t$ we have

$$\begin{aligned}\psi_t(Z(x), y) &= \varphi(Z(x), e^{t-1}y) = -(-1)^{i(\deg\ x)}\varphi(x, Z(e^{t-1}y))\\ &= -(-1)^{i(\deg\ x)}\varphi(x, e^{t-1}Z(y))\\ &= -(-1)^{i(\deg\ x)}\psi_t(x, Z(y))\end{aligned}$$

Hence, the homomorphism in (3.3) defines an injective map

$$\mathfrak{g}^e(0) \to \mathfrak{osp}(M_1) \times \mathfrak{osp}(\Pi(M_2)) \times \mathfrak{osp}(M_3) \times \mathfrak{osp}(\Pi(M_4)) \times \mathfrak{osp}(M_5) \times \cdots .$$

The fact that this map is surjective can be checked by direct computation. In particular, one should check that if $Z \in \mathfrak{gl}(m|2n)^e(0)_i$ satisfies $\psi_t(Z(x), y) = -(-1)^{i(\deg\ x)}\psi_t(x, Z(y))$ for all homogeneous $x, y \in M_t$ and for all $t \in \mathbb{Z}_+$, then $Z \in \mathfrak{osp}(m|2n)$. □

4 Good $\mathbb{Z}$-gradings

4.1 Good $\mathbb{Z}$-gradings of Lie superalgebras

Definition 4.1. Let $\mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}(j)$ be a $\mathbb{Z}$-graded Lie superalgebra. An element $e \in \mathfrak{g}_{\bar{0}}(2)$ is called *good* if the following properties hold:

$$\text{ad } e : \mathfrak{g}(j) \to \mathfrak{g}(j + 2) \text{ is injective for } j \leq -1; \tag{8}$$

$$\text{ad } e : \mathfrak{g}(j) \to \mathfrak{g}(j + 2) \text{ is surjective for } j \geq -1. \tag{9}$$

A $\mathbb{Z}$-grading of a Lie superalgebra $\mathfrak{g}$ is called *good* if it admits a good element.

Clearly, (8) is equivalent to

$$Ker(\text{ad } e) = \mathfrak{g}^e \subset \mathfrak{g}_{\geq}, \tag{10}$$

and (9) is equivalent to

$$\mathfrak{g}_+ \subset Im(\text{ad } e) = [e, \mathfrak{g}]. \tag{11}$$

Lemma 4.2. *Let $\mathfrak{g}$ be a $\mathbb{Z}$-graded Lie superalgebra. If $\mathfrak{g}$ is a basic Lie superalgebra or a semisimple Lie algebra, or if $\mathfrak{g} = \mathfrak{gl}(m|n)$ or $\mathfrak{sl}(n|n)$ and $\mathfrak{Z}(\mathfrak{g}_{\bar{0}}) \subset \mathfrak{g}(0)$, then for $e \in \mathfrak{g}_{\bar{0}}(2)$ conditions (8)-(11) are equivalent.*

Proof. If $\mathfrak{g} = \mathfrak{gl}(m|n)$, we may restrict to $\mathfrak{sl}(m|n)$ since $\mathfrak{Z}(\mathfrak{g}_{\bar{0}}) \subset \mathfrak{g}(0)$. By Lemma 2.7, the proof of [6, Theorem 1.3] proves (8) $\Leftrightarrow$ (9) $\square$

Lemma 4.3. *Let $\mathfrak{g}$ be a $\mathbb{Z}$-graded Lie superalgebra. If $\mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}(j)$ is a good grading for an element $e \in \mathfrak{g}_{\bar{0}}(2)$, then the induced grading of $\mathfrak{g}_{\bar{0}}$ is a good grading for e. Moreover, if $\mathfrak{g}$ is a basic Lie superalgebra, then $\mathfrak{Z}(\mathfrak{g}) \subset \mathfrak{g}(0)$, $\mathfrak{Z}(\mathfrak{g}_{\bar{0}}) \subset \mathfrak{g}_{\bar{0}}(0)$ and $e \in \mathfrak{g}'_{\bar{0}}(2) = \mathfrak{g}_{\bar{0}}(2)$.*

Proof. Now ad e preserves parity since $e \in \mathfrak{g}_{\bar{0}}$, i.e. $(\text{ad } e)(\mathfrak{g}_{\bar{0}}) \subset \mathfrak{g}_{\bar{0}}$, $(\text{ad } e)(\mathfrak{g}_{\bar{1}}) \subset \mathfrak{g}_{\bar{1}}$. So the map ad $e : \mathfrak{g}(j) \to \mathfrak{g}(j+2)$ is surjective (resp. injective) if and only if the maps ad $e : \mathfrak{g}_{\bar{0}}(j) \to \mathfrak{g}_{\bar{0}}(j+2)$, ad $e : \mathfrak{g}_{\bar{1}}(j) \to \mathfrak{g}_{\bar{1}}(j+2)$ are both surjective (resp. injective). In particular, if the $\mathbb{Z}$-grading of $\mathfrak{g}$ is a good grading for e, then the induced grading of $\mathfrak{g}_{\bar{0}}$ is a good grading for e. The second claim now follows from Lemma 4.2 since $\mathfrak{Z}(\mathfrak{g}) \subset \mathfrak{g}^e \subset \mathfrak{g}_{\geq 0}$ and $(\mathfrak{Z}(\mathfrak{g}) \cap \mathfrak{g}_+) \subset (\mathfrak{Z}(\mathfrak{g}) \cap Im(\text{ad } e)) = 0$. $\square$

The proofs of the following lemmas are straightforward.

Lemma 4.4. *Let $\mathfrak{a} = \mathfrak{a}' \times \mathfrak{Z}(\mathfrak{a})$ be a reductive Lie algebra. Then $\mathfrak{a} = \oplus_{j\in\mathbb{Z}}\mathfrak{a}(j)$ is good $\mathbb{Z}$-grading for $e \in \mathfrak{a}(2)$ if and only if $\mathfrak{Z}(\mathfrak{a}) \subset \mathfrak{a}(0)$, $e \in \mathfrak{a}'(2)$ and $\mathfrak{a}' = \oplus_{j\in\mathbb{Z}}\mathfrak{a}'(j)$ is a good $\mathbb{Z}$-grading for e.*

Lemma 4.5. *Let $\mathfrak{a}$ be a semisimple Lie algebra, and let $\mathfrak{a} = I_1 \oplus \cdots \oplus I_k$ be the unique decomposition of $\mathfrak{a}$ into ideals such that I_i are simple as Lie algebras. Let $e \in \mathfrak{a}(2)$, and write $e = e_1 + \cdots + e_k$ where $e_i \in I_i$. Then $\mathfrak{a} = \oplus_{j\in\mathbb{Z}}\mathfrak{a}(j)$ is a good (Dynkin) grading for e if and only if for each i the induced grading $I_i = \oplus_{j\in\mathbb{Z}}I_i(j)$ is a good (Dynkin) grading for the element e_i.*

4.2 Good $\mathbb{Z}$-gradings of basic Lie superalgebras

Let $\mathfrak{g}$ be a basic Lie superalgebra, $\mathfrak{gl}(m|n)$ or $\mathfrak{sl}(n|n)$. Then $\mathfrak{g}_{\bar{0}} = \mathfrak{g}'_{\bar{0}} \oplus \mathfrak{Z}(\mathfrak{g}_{\bar{0}})$ is a reductive Lie algebra. A $\mathbb{Z}$-grading is called a *Dynkin grading* if it is defined by ad h, where h belongs to an $\mathfrak{sl}_2$-triple $\mathfrak{s} = \{e, f, h\}$ satisfying $[e, f] = h$, $[h, e] = 2e$ and $[h, f] = -2f$. By $\mathfrak{sl}_2$ theory, e is a good element for this $\mathbb{Z}$-grading. Hence, all Dynkin gradings are good. Moreover, every nilpotent even element has a good grading. Indeed, we can apply the Jacobson-Morosov Theorem to $\mathfrak{g}'_{\bar{0}}$, since elements of $\mathfrak{Z}(\mathfrak{g}_{\bar{0}})$ are semisimple.

Theorem 4.6. *Let $\mathfrak{g}$ be a basic Lie superalgebra, $\mathfrak{gl}(m|n)$ or $\mathfrak{sl}(n|n)$. Let $\mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}(j)$ be a good $\mathbb{Z}$-grading for $e \in \mathfrak{g}_{\bar{0}}(2)$, and let $\mathfrak{s} = \{e, f, h\}$ be an $\mathfrak{sl}_2$-triple containing e. Then $\mathfrak{g}^{\mathfrak{s}} \subset \mathfrak{g}(0)$ and $\mathfrak{g}^{\mathfrak{s}}_{\bar{0}} \subset \mathfrak{g}_{\bar{0}}(0)$.*

Proof. By Lemma 4.3, the induced grading of $\mathfrak{g}_{\bar{0}}$ is a good grading for e and $e \in \mathfrak{g}'_{\bar{0}}$. Since $\mathfrak{g}'_{\bar{0}}$ is semisimple and e is nilpotent, by the Jacobson-Morosov Theorem there exists an $\mathfrak{sl}_2$-triple $\mathfrak{s} = \{e, f, h\} \subset \mathfrak{g}'_{\bar{0}}$ containing e. We have that $\mathfrak{g}^{\mathfrak{s}} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}^{\mathfrak{s}}(j)$. Now $\mathfrak{g}^{\mathfrak{s}} \subset \mathfrak{g}^e$ since $e \in \mathfrak{s}$ and $\mathfrak{g}^e \subset \mathfrak{g}_{\geq}$ by Lemma 4.2, hence $\mathfrak{g}^{\mathfrak{s}}(j) = 0$ for all $j \leq -1$.

Now $\mathfrak{g}$ is a finite dimensional module under the adjoint action of the $\mathfrak{sl}_2$ subalgebra generated by $\mathfrak{s}$. By $\mathfrak{sl}_2$ representation theory,

$$\mathfrak{g} = \mathfrak{g}^f \oplus Im(\text{ad } e). \tag{$*$}$$

Since $\mathfrak{g}^{\mathfrak{s}} = \mathfrak{g}^e \cap \mathfrak{g}^f$, it follows from $(*)$ that $\mathfrak{g}^{\mathfrak{s}} \cap Im(\text{ad } e) = \{0\}$. But then Lemma 4.2 implies that $\mathfrak{g}^{\mathfrak{s}}(j) = 0$ for all $j \geq 1$. Therefore $\mathfrak{g}^s \subset \mathfrak{g}(0)$. The second claim follows from the fact that the induced grading of $\mathfrak{g}_{\bar{0}}$ is also a good grading for e. $\square$

If $e \in \mathfrak{g}'_{\bar{0}}(2)$, $e \neq 0$, then [6, Lemma 1.1] gives the existence of $h \in \mathfrak{g}'_{\bar{0}}(0)$ and $f \in \mathfrak{g}'_{\bar{0}}(-2)$ such that $\mathfrak{s} = \{e, f, h\}$ is an $\mathfrak{sl}_2$-triple.

Corollary 4.7. *Let $\mathfrak{g}$ be a basic Lie superalgebra, $\mathfrak{g} \neq \mathfrak{psl}(n|n)$, or let $\mathfrak{g} = \mathfrak{gl}(m|n)$. Let $\mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}(j)$ be a good $\mathbb{Z}$-grading for $e \in \mathfrak{g}_{\bar{0}}(2)$ defined by $H \in \mathfrak{g}_{\bar{0}}(0)$. If $\mathfrak{s} = \{e, f, h\}$ is an $\mathfrak{sl}_2$-triple with $f \in \mathfrak{g}'_{\bar{0}}(-2)$ and $h \in \mathfrak{g}'_{\bar{0}}(0)$ given by [6, Lemma 1.1], then $z := H - h \in \mathfrak{Z}(\mathfrak{g}^{\mathfrak{s}})_{\bar{0}}$. In particular, if $\mathfrak{Z}(\mathfrak{g}^{\mathfrak{s}})_{\bar{0}} = \{0\}$, then the Dynkin grading is the only good grading for which e is a good element.*

We note that $H, h \in \mathfrak{g}_{\bar{0}}(0)$ are commuting semisimple elements, so we may choose our Cartan subalgebra to contain them. In particular, we see that all good gradings for an element e can be described (up to equivalence) by adding a semisimple element $z \in \mathfrak{Z}(\mathfrak{g}^{\mathfrak{s}})_{\bar{0}}$ to the element h of an $\mathfrak{sl}_2$-triple $\mathfrak{s} = \{e, f, h\}$.

The proof of the following lemma is the same as for Lie algebras [6]. Similarly, the theorem [6, Theorem 1.4] and its corollaries can be extended to basic Lie superalgebras.

Lemma 4.8. *Let $\mathfrak{g}$ be a basic Lie superalgebra and let $\Pi \subset \Delta_{\geq}$ be given by Lemma 2.8. If $\mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}(j)$ is a good $\mathbb{Z}$-grading, then $\Pi = \Pi_0 \sqcup \Pi_1 \sqcup \Pi_2$ and $\Pi_{\bar{0}} = \Pi_{\bar{0},0} \sqcup \Pi_{\bar{0},1} \sqcup \Pi_{\bar{0},2}$. In particular, all good gradings are among those defined by deg$(\alpha_i) = -$deg$(-\alpha_i) = 0, 1$, or 2, $i = 1, \ldots, n$ for some choice of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_n\}$.*

4.3 Richardson elements

Let $\mathfrak{g}$ be a basic Lie superalgebra. Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$, with nilradical $\mathfrak{n}$. We call an even or odd element $e \in \mathfrak{n}$ a *Richardson element* if $[\mathfrak{p}, e] = \mathfrak{n}$. For a finite-dimensional simple Lie algebra $\mathfrak{g}$ this definition is equivalent to the standard definition. If G is the adjoint group of $\mathfrak{g}$, then an element e in the nilradical $\mathfrak{n}$ is called a Richardson element for the Lie algebra $\mathfrak{p}$ of the parabolic subgroup $P \subset G$ if the orbit Pe is open dense in $\mathfrak{n}$ [4, 6].

Recall that $\mathfrak{g}_{\geq}$ is a parabolic subalgebra of $\mathfrak{g}$ with nilradical $\mathfrak{g}_{+}$.

Lemma 4.9. *Let $\mathfrak{g} = \oplus_{j\in 2\mathbb{Z}}\mathfrak{g}(j)$ be an even $\mathbb{Z}$-grading and let $\mathfrak{g}_{\geq}$ be the corresponding parabolic subalgebra of $\mathfrak{g}$. Let $e \in \mathfrak{g}_{\bar{0}}(2)$. Then e is a Richardson element for $\mathfrak{g}_{\geq}$ if and only if e is good.*

Proof. Since $e \in \mathfrak{g}(2)$ and the grading is even, $[\mathfrak{g}_{-}, e] \subset \mathfrak{g}_{\leq}$. Clearly, $[\mathfrak{g}_{\geq}, e] \subset \mathfrak{g}_{+}$. By Lemma 4.2, the grading is good for e if and only if $\mathfrak{g}_{+} \subset [\mathfrak{g}, e]$. Hence, the grading is good for e if and only if $[\mathfrak{g}_{\geq}, e] = \mathfrak{g}_{+}$. □

It is important to note that a parabolic subalgebra of a basic Lie superalgebra does not necessarily have a Richardson element. In particular, a Borel subalgebra of $\mathfrak{sl}(m|n)$ for $m \neq n \pm 1$ does not have a Richardson element.

4.4 Extending good $\mathbb{Z}$-gradings of $\mathfrak{g}_{\bar{0}}$

Given a basic Lie superalgebra $\mathfrak{g}$, it is natural to ask: which good $\mathbb{Z}$-gradings of $\mathfrak{g}_{\bar{0}}$ *extend* to good $\mathbb{Z}$-gradings of $\mathfrak{g}$, and to what extent is such an extension determined by the $\mathbb{Z}$-grading of $\mathfrak{g}_{\bar{0}}$.

The following lemma is easy to prove.

Lemma 4.10. *Let $\mathfrak{g}$ be a Lie superalgebra. If $H, H' \in \mathfrak{g}_{\bar{0}}$ are such that ad H and ad H' define $\mathbb{Z}$-gradings of $\mathfrak{g}$ for which the induced gradings of $\mathfrak{g}_{\bar{0}}$ coincide, then $H - H' \in \mathfrak{Z}(\mathfrak{g}_{\bar{0}})$.*

The following lemma is a corollary of Lemma 4.10.

Lemma 4.11. *Let $\mathfrak{g}$ be a basic Lie superalgebra such that $\mathfrak{Z}(\mathfrak{g}_{\bar{0}}) = 0$ and all derivations of $\mathfrak{g}$ are inner, i.e. $\mathfrak{osp}(m|2n) : m \neq 2n+2$, $F(4)$, $G(3)$, $D(2,1,\alpha)$. Then a $\mathbb{Z}$-grading of $\mathfrak{g}_{\bar{0}}$ has a unique extension to a $\mathbb{Z}$-grading of $\mathfrak{g}$.*

Remark 4.12. A good $\mathbb{Z}$-grading of $\mathfrak{g}_{\bar{0}}$ need not have a good extension to $\mathfrak{g}$. The main theorem will provide counterexamples. See Example 7.3.

Lemma 4.13. *A Dynkin grading of $\mathfrak{g}_{\bar{0}}$ has an extension to a Dynkin grading of $\mathfrak{g}$.*

Proof. Let $\mathfrak{g}_{\bar{0}} = \sum_{j\in\mathbb{Z}} \mathfrak{g}_{\bar{0}}(j)$ be a Dynkin grading defined by ad h with good element $e \in \mathfrak{g}_{\bar{0}}(2)$ such that $\mathfrak{s} = \{e, f, h\} \subset \mathfrak{g}_{\bar{0}}$ is an $\mathfrak{sl}_2$-triple. Then $\mathfrak{g}$ is a finite dimensional module under the adjoint action of the $\mathfrak{sl}(2)$ subalgebra generated by $\mathfrak{s}$. Hence it decomposes into a direct sum of irreducible modules. The action ad h defines a $\mathbb{Z}$-grading of $\mathfrak{g}$ for which e is a good element. ☐

Remark 4.14. In the case that $\mathfrak{Z}(\mathfrak{g}_{\bar{0}}) \neq 0$ it is possible that a good $\mathbb{Z}$-grading of $\mathfrak{g}_{\bar{0}}$ has more then one extension to a good $\mathbb{Z}$-grading of $\mathfrak{g}$. See Example 7.1.

5 Good $\mathbb{Z}$-gradings for the exceptional basic Lie superalgebras

Theorem 5.1. *All good $\mathbb{Z}$-gradings of the exceptional Lie superalgebras $F(4)$, $G(3)$ and $D(2,1,\alpha)$ are Dynkin gradings.*

Proof. Let $\mathfrak{g}$ be one of the exceptional basic Lie superalgebras, $F(4)$, $G(3)$ or $D(2,1,\alpha)$. We see from Table 1 that the center of $\mathfrak{g}_{\bar{0}}$ is trivial. Let $\mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}(j)$ be a good $\mathbb{Z}$-grading with good element $e \in \mathfrak{g}(2)$. The induced grading of each simple ideal of $\mathfrak{g}_{\bar{0}}$ is a good grading for e by Lemma 4.3 and Lemma 4.5. The grading of $\mathfrak{g}_{\bar{0}}$ is a Dynkin grading if and only if the induced grading of each simple ideal is a Dynkin grading. All derivations of $\mathfrak{g}$ are inner [9, 17]. Since $\mathfrak{Z}(\mathfrak{g}_{\bar{0}}) = 0$, an extension of a $\mathbb{Z}$-grading of $\mathfrak{g}_{\bar{0}}$ to a $\mathbb{Z}$-grading of $\mathfrak{g}$ is unique, by Lemma 4.11. A Dynkin grading of $\mathfrak{g}_{\bar{0}}$ has an extension to a Dynkin grading of $\mathfrak{g}$. Hence, if the induced grading of $\mathfrak{g}_{\bar{0}}$ is a Dynkin grading then the $\mathbb{Z}$-grading of $\mathfrak{g}$ is also Dynkin. If $\mathfrak{g} = G(3)$ then $\mathfrak{g}_{\bar{0}} = G_2 \times \mathfrak{sl}(2)$. If $\mathfrak{g} = D(2,1,\alpha)$ then $\mathfrak{g}_{\bar{0}} = \mathfrak{sl}(2) \times \mathfrak{sl}(2) \times \mathfrak{sl}(2)$. It was shown in [6] that every good $\mathbb{Z}$-grading of G_2 and of $\mathfrak{sl}(2)$ is a Dynkin grading. Hence, all good $\mathbb{Z}$-gradings of $G(3)$ and of $D(2,1,\alpha)$ are Dynkin gradings.

If $\mathfrak{g} = F(4)$, then $\mathfrak{g}_{\bar{0}} = \mathfrak{so}(7) \times \mathfrak{sl}(2)$. By [6], the only non-Dynkin gradings of $\mathfrak{so}(7)$ correspond to the nilpotent element with partition $(3,3,1)$. The induced grading of $\mathfrak{sl}(2)$ is a good Dynkin grading. By Lemma 2.1, there exists a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}_{\bar{0}}(0)$. By Lemma 2.3, the root space decomposition is compatible with the $\mathbb{Z}$-grading. We fix the following set of simple roots for $F(4)$.

$$\begin{tikzcd} \underset{\alpha_1}{\bigcirc} \arrow[r, shift left, "-1"] & \underset{\alpha_2}{\bigcirc} \arrow[l, shift left, "-1"] \arrow[r, shift left, "-1"] & \underset{\alpha_3}{\bigcirc} \arrow[l, shift left, "-2"] \arrow[r, shift left, "-1"] & \underset{\alpha_4}{\bigotimes} \arrow[l, shift left, "1"] \end{tikzcd}$$

Then $\{\alpha_1, \alpha_2, \alpha_3\}$ is a set of simple roots for the simple ideal isomorphic to $\mathfrak{so}(7)$. The highest root $\theta = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$ is a root for the simple ideal isomorphic to $\mathfrak{sl}(2)$. This implies that $\mathrm{Deg}(\theta) = \pm 2$. The nilpotent element of $\mathfrak{so}(7)$ corresponding to the partition $(3,3,1)$ is (up to conjugacy) $e_1 = X_1 + X_2$ where $X_1 \in \mathfrak{g}_{\alpha_1}$, $X_2 \in \mathfrak{g}_{\alpha_2}$. The Dynkin grading for e_1 is $[\deg(\alpha_1), \deg(\alpha_2), \deg(\alpha_3)] = [2,2,-2]$, and the non-Dynkin gradings are $[2,2,-1]$ and $[2,2,-3]$. For the non-Dynkin gradings, we have that $\mathrm{Deg}(\theta) = 3 + 2\mathrm{Deg}(\alpha_4)$ and $\mathrm{Deg}(\theta) = -3 + 2\mathrm{Deg}(\alpha_4)$, respectively. Since $\mathrm{Deg}(\alpha_4) \in \mathbb{Z}$ this implies $\mathrm{Deg}(\theta)$ is odd, which is impossible since $\mathrm{Deg}(\theta) = \pm 2$. ☐

6 Good $\mathbb{Z}$-gradings for $\mathfrak{psl}(2|2)$

We adopt the notation of Lemma 2.6. The following lemma can proven by explicitly computing $\mathfrak{g}^e$.

Lemma 6.1. *The $\mathbb{Z}$-grading of $\mathfrak{psl}(2|2)$ defined by $m \in \mathbb{Z}$, $p, q \in \{0, 2\}$ and $(a : b), (c : d) \in \mathbb{P}^2$ satisfying $(a : b) \neq (c : d)$ is a good grading for the element $e = rE_{12} + sE_{34}$ if and only if $p = 0 \Leftrightarrow r = 0$, $q = 0 \Leftrightarrow s = 0$ and $0 \leq m \leq p + q$.*

7 Good $\mathbb{Z}$-gradings for $\mathfrak{gl}(m|n)$

In this section we classify the good $\mathbb{Z}$-gradings of $\mathfrak{g} = \mathfrak{gl}(m|n)$. The good $\mathbb{Z}$-gradings of $\mathfrak{sl}(m|n) : m \neq n$ and $\mathfrak{psl}(n|n) : n \neq 2$ are uniquely induced from good $\mathbb{Z}$-gradings of $\mathfrak{g} = \mathfrak{gl}(m|n)$ since $\mathfrak{z}(\mathfrak{g}_{\bar{0}}) \subset \mathfrak{g}(0)$. See Lemma 2.5). To describe these gradings we generalize the definition of a pyramid given in [3, 6] to the Lie superalgebra $\mathfrak{gl}(m|n)$.

A pyramid P is a finite collection of boxes of size 2×2 in the upper half plane which are centered at integer coordinates, such that for each $j = 1, \ldots, N$, the second coordinates of the j^{th} row equal $2j - 1$ and the first coordinates of the j^{th} row form an arithmetic progression $f_j, f_j + 2, \ldots, l_j$ with difference 2, such that the first row is centered at $(0, 0)$, i.e. $f_1 = -l_1$, and

$$f_j \leq f_{j+1} \leq l_{j+1} \leq l_j \quad \text{for all } j. \tag{12}$$

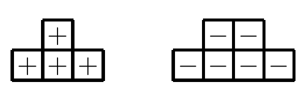
Each box of P has even or odd parity. We say that P has *size* $(m|n)$ if P has exactly m even boxes and n odd boxes.

Fix $m, n \in \mathbb{Z}_+$ and let (p, q) be a partition of $(m|n)$. Let $r = \psi(p, q) \in Par(m + n)$ be the total ordering of the partitions p and q which satisfies: if $p_i = q_j$ for some i, j then $\psi(p_i) < \psi(q_j)$. We define $Pyr(p, q)$ to be the set of pyramids which satisfy the following two conditions: (1) the j^{th} row of a pyramid $P \in Pyr(p, q)$ has length r_j; (2) if $\psi^{-1}(r_j) \in p$ (resp. $\psi^{-1}(r_j) \in q$) then all boxes in the j^{th} row have even (resp. odd parity) and we mark these boxes with a "+" (resp. "−" sign).

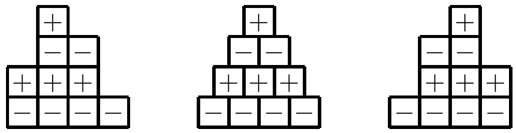
Corresponding to each pyramid $P \in Pyr(p, q)$ we define a nilpotent element $e(P) \in \mathfrak{g}_{\bar{0}}$ and semisimple element $h(P) \in \mathfrak{g}_{\bar{0}}$, as follows. Recall $\mathfrak{gl}(m|n) = \mathrm{End}(V_0 \oplus V_1)$. Fix a basis $\{v_1, \ldots, v_m\}$ of V_0 and $\{v_{m+1}, \ldots, v_{m+n}\}$ of V_1. Label the even (resp. odd) boxes of P by the basis vectors of V_0 (resp. V_1). Define an endomorphism $e(P)$ of $V_0 \oplus V_1$ as acting along the rows of the pyramid, i.e. by sending a basis vector v_i to the basis vector which labels the box to the right of the box labeled by v_i or to zero if it has no right neighbor. Then $e(P)$ is nilpotent and corresponds to the partition (p, q). Since $e(P)$ does not depend the choice of P in $Pyr(p, q)$, we may denote it by $e_{p,q}$. Moreover, $e_{p,q} \in \mathfrak{g}_{\bar{0}}$ because boxes in the same row have the same parity.

Define $h(P)$ to be the $(m+n)$-diagonal matrix where the i^{th} diagonal entry is the first coordinate of the box labeled by the basis vector v_i. Then $h(P)$ defines a $\mathbb{Z}$-grading of $\mathfrak{g}$ for which $e_{p,q} \in \mathfrak{g}(2)$. Let $P_{p,q}$ denote the symmetric pyramid from $Pyr(p, q)$. Then $h(P_{p,q})$ defines a Dynkin grading for $e_{p,q}$, and $P_{p,q}$ is called the *Dynkin pyramid* for the partition $(p|q)$.

Example 7.1. *Let $\mathfrak{g} = \mathfrak{gl}(4|6)$ and consider the partitions $p = (3, 1)$ and $q = (4, 2)$. The Dynkin grading of $\mathfrak{g}_{\bar{0}} = \mathfrak{gl}(4) \times \mathfrak{gl}(6)$ for the partition (p, q) corresponds to the following symmetric pyramids.*



There are pyramids in $Pyr(p,q)$ for which the induced grading of $\mathfrak{g}_{\bar{0}}$ is the one given above, and these correspond to good $\mathbb{Z}$-gradings. They are represented by the following pyramids:



Theorem 7.2. *Let $\mathfrak{g} = \mathfrak{gl}(m|n)$, and let (p,q) be a partition of $(m|n)$. If P is a pyramid from $Pyr(p,q)$, then the pair $(h(P), e_{p,q})$ is good. Moreover, every good grading for $e_{p,q}$ is of the form $(h(P), e_{p,q})$ for some pyramid $P \in Pyr(p,q)$.*

Proof. By Lemma 3.1, this can be proven using the same method as for $\mathfrak{gl}(m+n)$ given in [6]. It is easy to see from [6, Figures 1-3] that if $e = e_{p,q}$ and $P \in Pyr(p,q)$, then the eigenvalues of ad $h(P)$ on $\mathfrak{g}^e$ are nonnegative. Conversely, a good $\mathbb{Z}$-grading for $e_{p,q}$ is defined by the eigenvalues of $\mathrm{ad}((h(P_{p,q})+z)$ where $z \in \mathfrak{Z}(\mathfrak{g}^{\mathfrak{s}})_{\bar{0}}$ is a diagonal matrix with integer entries and $\mathfrak{s} = \{e_{p,q}, h(P_{p,q}), f\}$ is an $\mathfrak{sl}_2$-triple (see Section 4.2). It is easy to see from [6, Figures 1-3] that the condition $z \in \mathfrak{Z}(\mathfrak{g}^{\mathfrak{s}})_{\bar{0}}$ implies that the diagonal entries of z must be constant along each row of the pyramid and equal on rows of the same length. Moreover, condition (12) must be satisfied in order for the eigenvalues of $\mathrm{ad}(h(P_{p,q})+z)$ on $\mathfrak{g}^e$ to be nonnegative. So $h(P_{p,q})+z = h(P)$ for some pyramid $P \in Pyr(p,q)$. $\square$

Example 7.3. *Let $\mathfrak{g} = \mathfrak{gl}(4|6)$ and consider the partitions $p = (3,1)$ and $q = (4,2)$. The following pyramids represent a good $\mathbb{Z}$-grading of $\mathfrak{g}_{\bar{0}}$ for which there is no good $\mathbb{Z}$-grading of $\mathfrak{g}$ with this induced good $\mathbb{Z}$-grading of $\mathfrak{g}_{\bar{0}}$.*

8 Good $\mathbb{Z}$-gradings for $\mathfrak{osp}(m|2n)$

In this section we classify good $\mathbb{Z}$-gradings for $\mathfrak{g} = \mathfrak{osp}(m,2n)$. Recall that $\mathfrak{g}_{\bar{0}} = \mathfrak{so}(m) \times \mathfrak{sp}(2n)$. To describe these gradings we define an orthosymplectic pyramid, generalizing the definition of orthogonal and symplectic pyramids as defined in [3, 6].

Given a partition p, we let $J_p = \{p_1 > \cdots > p_a\}$ be the set of distinct nonzero parts of p. We write $p = (p_1^{m_{p_1}}, \ldots, p_a^{m_{p_a}})$, where m_{p_i} is the multiplicity of p_i in p. A partition is called orthogonal (resp. symplectic) if m_{p_i} is even for even (resp. odd) p_i. We say that a partition $(p|q)$ of $(m|2n)$ is orthosymplectic if p is an orthogonal partition of m and q is a symplectic partition of $2n$.

Let $(p|q)$ be an orthosymplectic partition of $(m|2n)$. Let $r \in Par(m+2n)$ be the total ordering of the partitions p and q. Let $J_r = \{r_1 > \cdots > r_b\}$ be the set of distinct nonzero parts of r. Write $r = (r_1^{m_1+n_1}, \ldots, r_b^{m_b+n_b})$ where $p = (r_1^{m_1}, \ldots, r_b^{m_b})$ and $q = (r_1^{n_1}, \ldots, r_b^{n_b})$.

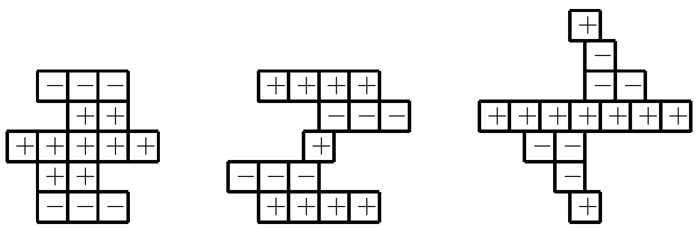
We define the *orthosymplectic Dynkin pyramid* for $(p|q)$ as follows. It is a finite collection of boxes of size 2×2 in the plane centered at integer coordinates: $(i, 2j)$ for m odd and $(i, 2j-1)$ for m even. It is centrally symmetric about $(0,0)$. We describe how to place the boxes in the upper half plane. The boxes in lower half plane are obtained by the central symmetry.

If m is even, then the zeroth row is empty. If m is odd, let r_k be the largest part of p occurring with odd multiplicity. Put r_k boxes in the zeroth row in the columns $1-r_k, 3-r_k, \ldots, r_k-1$, and

remove one part of r_k from the partition. Now p has an even number of parts occurring with odd multiplicity. Denote these by $c_1 > d_1 > \cdots > c_N > d_N$.

We add boxes inductively to the next row in the upper half plane as follows. Let r_j be the largest part remaining in the partition r. If m_j is odd, then $r_j = c_i$ for some i. We add an "even skew-row" of length $\frac{c_i+d_i}{2}$ of even parity boxes in the columns $1-d_i, 3-d_i, \ldots, c_i-1$, and then remove c_i and d_i from the partition. Next we add $\lfloor \frac{m_j}{2} \rfloor$ rows of length r_j of even parity boxes in the columns $1-r_j, 3-r_j, \ldots, r_j-1$. If n_j is odd, we then add an "odd skew-row" of length $\frac{r_j}{2}$ of odd parity boxes in the columns $1, \ldots, r_j-1$. Finally we add $\lfloor \frac{n_j}{2} \rfloor$ rows of length r_j of odd parity boxes in the columns $1-r_j, 3-r_j, \ldots, r_j-1$, and remove $r_j^{m_j+n_j}$ from the partition. We label the even boxes with the symbol "+" and the odd boxes with the symbol "−".

Example 8.1. $\mathfrak{osp}(9|6)$. *The pyramids for the partitions* $(5,3,1|3,3)$, $(4,4,1|6)$, $(7,1,1|4,2)$ *are:*



We define a nilpotent element $e_{p,q} \in \mathfrak{g}_{\bar{0}}$ and semisimple element $h_{p,q} \in \mathfrak{g}_{\bar{0}}$ as in [3]. Let φ be a non-degenerate supersymmetric bilinear form on $V = V_0 \oplus V_1$, so that V_0 and V_1 are orthogonal and the restriction to V_0 is symmetric while the restriction to V_1 is skew-symmetric. Let $k = \lfloor \frac{m}{2} \rfloor$. We take the standard basis $\{v_0, v_1, \ldots, v_k, v_{-1}, \ldots, v_{-k}\}$ of $V_{\bar{0}}$ and $\{v_{k+1}, \ldots, v_{k+n}, v_{-(k+1)}, \ldots, v_{-(k+n)}\}$ of $V_{\bar{1}}$, which for $i, j > 0$ satisfies $\varphi(v_0, v_0) = 2$, $\varphi(v_0, v_{\pm j}) = 0$, $\varphi(v_i, v_j) = \varphi(v_{-i}, v_{-j}) = 0$, and $\varphi(v_i, v_{-j}) = \delta_{ij}$. We omit v_0 if $m = 2k$.

We write $E_{i,j}$ for the matrix with a 1 in the (i,j) place and zeros elsewhere. The following matrices give a Chevalley basis for $\mathfrak{osp}(m|2n)_{\bar{0}} = \mathfrak{so}(m) \times \mathfrak{sp}(2n)$ (omitting the first set if $m = 2k$)

$$\begin{gathered}\{2E_{i,0} - E_{0,-i}, E_{0,i} - 2E_{-i,0}\}_{1 \le i \le k} \cup \{E_{i,-j} - E_{j,-i}, E_{-j,i} - E_{-i,j}\}_{1 \le i < j \le k} \\ \cup \{E_{i,j} - E_{-j,-i}\}_{1 \le i,j \le k} \cup \{E_{i,j} - E_{-j,-i}\}_{k+1 \le i,j \le k+n} \\ \cup \{E_{i,-i}, E_{-i,i}\}_{k+1 \le i \le k+n} \cup \{E_{i,-j} + E_{j,-i}, E_{-i,j} + E_{-j,i}\}_{k+1 \le i < j \le k+n}.\end{gathered}$$

Define $\sigma_{i,j} \in \{\pm 1\}$ to be the coefficient of $E_{i,j}$ of the unique element in this basis if it appears, or zero if no basis element involves $E_{i,j}$ [3].

Label the even boxes (resp. odd boxes) in the upper half plane $x, y > 0$ with the vectors $v_1, \ldots, v_k$ (resp. $v_{k+1}, \ldots, v_{k+n}$). The centrally symmetric box of the box labeled with v_i is labeled with v_{-i}. There is a box at $(0,0)$ if and only if m is odd, in which case we label this box with v_0.

Define $e_{p,q}$ to be the matrix $\sum_{i,j} \sigma_{i,j} E_{i,j}$, where the sum is over all pairs of boxes B_i, B_j in the orthosymplectic Dynkin pyramid satisfying one of the following:

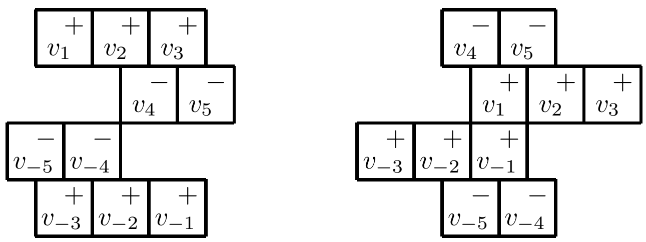
- $row(B_i) = row(B_j)$ and $col(B_i) = col(B_j) + 2$;
- $row(B_i) = -row(B_j)$ is an even skew-row in the upper half plane, $col(B_i) = 2, col(B_j) = 0$;
- $row(B_i) = -row(B_j)$ is an even skew-row in the upper half plane, $col(B_i) = 0, col(B_j) = -2$;

- $row(B_i) = -row(B_j)$ is an odd skew-row in the upper half plane, $col(B_i) = 1, col(B_j) = -1$,

where $row(B_i)$ (resp. $col(B_i)$) denotes the first (resp. second) coordinate of the box B_i. Then $e_{p,q}$ is a nilpotent element of $\mathfrak{osp}(m|2n)$ and corresponds to the partition $(p|q)$.

Define $h_{p,q}$ to be the $(m+2n)$-diagonal matrix whose eigenvalue on the vector v_i is equal to the first coordinate of the box labeled with this vector. Then $h_{p,q}$ defines a $\mathbb{Z}$-grading of $\mathfrak{g}$ for which $e_{p,q} \in \mathfrak{g}(2)$. This is the Dynkin grading for $e_{p,q}$.

Example 8.2. $\mathfrak{osp}(6,4)$. *The pyramids for the partitions* $(3,3|4)$, $(5,1|2,2)$ *are:*



Let

$$C(p) := \{p_i \in J_p \mid p_i \text{ is odd, } m_{p_i} = 2 \text{ and } p_i \notin J_q\} = \{p_1 > \cdots > p_{c(p)}\}$$

and

$$D(q) := \{q_j \in J_q \mid q_j \text{ is even, } m_{q_j} = 2 \text{ and } q_j \notin J_p\} = \{q_1 > \cdots > q_{d(q)}\}.$$

Define the diagonal matrices $z(s_1, \ldots, s_{c(p)}) \in \mathfrak{so}(m)$, with $s_i \in \mathbb{F}$, whose i^{th} diagonal entry is s_i if the basis vector lies in a box of $SP(p)$ in the (strictly) upper half-plane in a row corresponding to the part $p_i \in C(p)$, and is $-s_i$ if the basis vector lies in the centrally symmetric box, and all other entries are zero. Define the diagonal matrices $z(t_1, \ldots, t_{d(q)}) \in \mathfrak{sp}(2n)$, with $t_j \in \mathbb{F}$, whose j^{th} diagonal entry is t_j if the basis vector lies in a box of $SP(p)$ in the (strictly) upper half-plane in a row corresponding to the part $q_j \in D(q)$, and is $-t_j$ if the basis vector lies in the centrally symmetric box, and all other entries are zero.

Theorem 8.3. *Let* $\mathfrak{g} = \mathfrak{osp}(m|2n)$ *and let* (p,q) *be an orthosymplectic partition of* $(m|n)$.

If m=2k+1, the element $h_{p,q} + (z(s_1, \ldots, s_{c(p)}), z(t_1, \ldots, t_{d(q)}))$ *defines a good* $\mathbb{Z}$*-grading of* $\mathfrak{osp}(2k+1|2n)$ *for* $e_{p,q}$ *if and only if one of the following cases holds:*

(i) if $1 \notin C(p)$*, then* $s_i, t_j \in \{-1,0,1\}$ *for* $1 \le i \le c(p)$, $1 \le j \le d(q)$*, and for each pair* $p_k \in C(p)$, $q_l \in D(q)$ *satisfying* $p_k = q_l \pm 1$ *we must have* $|s_k - t_l| \le 1$*;*

(ii) if $1 \in C(p)$*, then* $s_i, t_j \in \{-1,0,1\}$ *for* $1 \le i \le c(p)-1$, $1 \le j \le d(q)$, $s_{c(p)} \in \mathbb{Z}$*, and for each pair* $p_k \in C(p)$, $q_l \in D(q)$ *satisfying* $p_k = q_l \pm 1$ *we must have* $|s_k - t_l| \le 1$*, and*

$$|s_{c(p)}| \;\le\; min\{p_{\alpha-1} - 1,\ q_\beta - 1,\ p_{c(p)-1} - |s_{c(p)-1}| - 1,\ q_{d(q)} - |t_{d(q)}| - 1\}.$$

If m=2k, the element $h_{p,q} + (z(s_1, \ldots, s_{c(p)}), z(t_1, \ldots, t_{d(q)}))$ *defines a good* $\mathbb{Z}$*-grading of* $\mathfrak{osp}(2k|2n)$ *for* $e_{p,q}$ *if and only if one of the following cases holds:*

(i) if $1 \notin C(p)$*, and* $C(p) \ne J_p$ *or* $D(q) \ne J_q$*, then* $s_i, t_j \in \{-1,0,1\}$ *for* $1 \le i \le c(p)$, $1 \le j \le d(q)$*, and for each pair* $p_k \in C(p)$, $q_l \in D(q)$ *satisfying* $p_k = q_l \pm 1$ *we must have* $|s_k - t_l| \le 1$*;*

(ii) *if* $1 \notin C(p)$ *and* $C(p) = J_p$, $D(q) = J_q$, *then either all* $s_i, t_j \in \{-1, 0, 1\}$ *for* $1 \leq i \leq c(p)$, $1 \leq j \leq d(q)$ *or all* $s_i, t_j \in \{-1/2, 1/2\}$, *and for each pair* $p_k \in C(p)$, $q_l \in D(q)$ *satisfying* $p_k = q_l \pm 1$ *we must have* $|s_k - t_l| \leq 1$*;*

(iii) *if* $1 \in C(p)$, *and* $C(p) \neq J_p$ *or* $D(q) \neq J_q$, *then* $s_i, t_j \in \{-1, 0, 1\}$ *for* $1 \leq i \leq c(p) - 1$, $1 \leq j \leq d(q)$, $s_{c(p)} \in \mathbb{Z}$, *and for each pair* $p_k \in C(p)$, $q_l \in D(q)$ *satisfying* $p_k = q_l \pm 1$ *we must have* $|s_k - t_l| \leq 1$, *and*

$$|s_{c(p)}| \ \leq \ min\{p_{\alpha-1} - 1, \ q_\beta - 1, \ p_{c(p)-1} - |s_{c(p)-1}| - 1, \ q_{d(q)} - |t_{d(q)}| - 1\}.$$

(iv) *if* $1 \in C(p)$ *and* $C(p) = J_p$, $D(q) = J_q$, *then either all* $s_i, t_j \in \{-1, 0, 1\}$ *for* $1 \leq i \leq c(p) - 1$, $1 \leq j \leq d(q)$, $s_{c(p)} \in \mathbb{Z}$, *or all* $s_i, t_j \in \{-1/2, 1/2\}$ *and* $s_{c(p)} \in (1/2 + \mathbb{Z})$, *and for each pair* $p_k \in C(p)$, $q_l \in D(q)$ *satisfying* $p_k = q_l \pm 1$ *we must have* $|s_k - t_l| \leq 1$, *and*

$$|s_{c(p)}| \ \leq \ min\{p_{\alpha-1} - 1, \ q_\beta - 1, \ p_{c(p)-1} - |s_{c(p)-1}| - 1, \ q_{d(q)} - |t_{d(q)}| - 1\}.$$

Proof. Let $e = e_{p,q}$, $h = h_{p,q}$ and let $\mathfrak{s} = \{e, h, f\}$ be the corresponding $\mathfrak{sl}_2$-triple. As in the $\mathfrak{gl}(m|n)$ case, all good $\mathbb{Z}$-gradings for $e_{p,q}$ can be describe by the eigenvalues and eigenspaces of $\mathrm{ad}(h_{p,q} + z)$ for some semisimple element $z \in \mathfrak{Z}(\mathfrak{g}^s)$. Recall that $\mathfrak{g}_{\bar{0}} = \mathfrak{so}(m) \times \mathfrak{sp}(2n)$. The conditions on the diagonal matrix z which imply $\mathfrak{g}^e_{\bar{0}} \subset (\mathfrak{g}_{\bar{0}})_{\geq}$ where determined in [6]. So we only need to determine the additional conditions which imply $\mathfrak{g}^e_{\bar{1}} \subset (\mathfrak{g}_{\bar{1}})_{\geq}$. Now $\mathfrak{g}_{\bar{1}} \cong \mathrm{Hom}(V_0, V_1)$, so these conditions are the same as for the odd part of the $\mathfrak{gl}(m|n)$ case. In particular, for all i, j we must have $s_i - t_j \in \mathbb{Z}$ and $|s_i - t_j| \leq |p_i - q_j|$ where s_i (resp. t_j) corresponds to the partition part p_i (resp. q_j). $\square$

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